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TWO-VARIABLE ORTHOGONAL MATRICES FROM SBIBD

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We show that the incidence matrix of an symmetric balanced incomplete block design can be used to make a new two-variable orthogonal matrix, X, whose variables, when replaced by suitable entries/ numbers/ values which have moduli 1 and which have at least one 1 in each row and column give a matrix satisfying $XX^T = \omega I$, ω a real constant. We illustrate this by application to a family of symmetric balanced incomplete block designs.

Keywords: *Hadamard Matrices, Orthogonal Matrices, Cretan Matrices, Symmetric Balanced Incomplete Block Designs (SBIBD), Regular Hadamard Matrices, 05B20*

INTRODUCTION

This work, created except where noted, by the author, has arisen from explaining the mathematics associated with some *Cretan* matrices. *Cretan* matrices were first discussed during a conference in Crete in 2014 by N. A. Balonin, M. B. Sergeev and colleagues of the Saint Petersburg State University of Aerospace Instrumentation, 67, B. Morskaia St., 190000, St. Petersburg, Russian Federation [1, 2, 7, 8].

This work follows closely the joint work of the author with N.A. Balonin and M.B. Sergeev [3, 4, 5, 6].

Aim: Our aim in this study is to search for τ-variable orthogonal matrices, X, with maximal or high determinant. We use x_1, $x_2,\ldots,x_\tau$ as variables to be replaced by entries/ values/ numbers having modulus ≤ 1 and at least one 1 per row and column and which have maximum or high determinant. The entries/ values/ numbers can be negative. These we call these new matrices Cretan matrices. Hence the aim of the study is to find τ-variable orthogonal matrices which yield Cretan matrices which have maximum or high determinant.

Symmetric balanced incomplete block designs or (v; k;λ) - configurations or SBIBD (v; k; λ) are of considerable use and interest to image processing (compression, masking) and to statisticians undertaking medical or agricultural research. We use the usual SBIBD convention that v > 2k and k > 2λ.

We see from the La Jolla Repository of difference sets [10] that there exist (v; k; λ) difference sets for $v = 4t + 1; 4t; 4t - 1; 4t - 2$ which can be used to make circulant SBIBD(v; k; λ).

In this and future papers we use some names, definitions, notations differently to how we they have been have in the past [2]. This we hope, will cause less confusion, bring our nomenclature closer to common usage, conform for mathematical purists and clarify the similarities and differences between some matrices. We have chosen to use the word level, instead of value for the entries of a matrix, to conform to earlier writings [2, 7, 8]

Preliminary Definitions

Definition 1: A orthogonal matrix, X, has real entries and satisfies

$XX^T = X^TX = \omega I_n$

Where, I_n is the n × n identity matrix, and ω, the weight, is a constant real number.

Definition 2: X = (x_{ij}) of order n will be called a τ-variable orthogonal matrix, with variables x_1; x_2; ...; x_τ when it is orthogonal, satisfying $XX^T = \omega I$ and for which $\sum_{j=1}^{n}(xij)^2 = \omega$, ω a real constant, for all i and $\sum^n{}_{i=1} x_{ji}x_{ki} = 0$ for each distinct pair of distinct rows j and k. A similar condition holds for the columns of X.

In this paper we only study 2-variable orthogonal matrices written with the variables x and y. Cretan matrices are made by choosing appropriate entries/ numbers/ values/ levels for x = 1 and $|y| \le 1$, where at least one entry in each row and column is 1.

***Definition 3*:** (Cretan). A Cretan matrix, S = (s_{ij}), of order n, written as Cretan(n; τ) or CM(n; τ), is a matrix of levels, ie numbers/values/entries, satisfing the orthogonality equation

$SS^T = S^TS= \omega I_n$

Where, ω, called the weight, is a real constant.

Cretan(n; τ) or CM(n; τ), have τ levels, they are made from τ-variable orthogonal matrices by replacing the variables by appropriate real numbers with moduli ≤ 1, where at least one entry in each row and column is 1. A Cretan matrix is orthogonal.

After the variables have been replaced by feasible entries/values/numbers S, Cretan(n;τ) or CM(n;τ), are used to denote one-the-other. In this paper τ= 2.

Notation 1: S, a -variable matrix with variables x_1; x_2; ; x is used to form Cretan(n; τ; ω; j_1; j_2;... j_τ ;l_1: l_2;...l_τ; determinant). Where n is the order, is the number of dis-tinct variables or levels (counting $-x$ separately from x); j_1; j_2;..; j_τ is the number of occurrences of the variables x_1; x_2;...; x_τ if they occur the same number of times in each row and column, however as this mostly does not happen these values are just omitted; l_1,l_2,...,l_τ the total number of each variable in the whole matrix Cretan(n;τ) or CM(n;τ), and the determinant. After the variables have been replaced by feasible entries/values/numbers S, Cretan(n;τ) or CM(n;τ), are used to denote one-the-other.

A Cretan matrix is orthogonal. It may be used to nd some real matrices with entries ≤ 1 which have with maximal or high determinant. In this paper τ= 2.

PRELIMINARY DEFINITIONS AND RESULTS: SBIBD

Definition 4 (Incidence Matrix): For the purposes of this paper we will consider an SBIBD(v; k; λ), B, to be a v ×v matrix, with entries 0 and 1, k ones per row and column, and the inner product of distinct pairs of rows and/or columns to be. This is called the incidence matrix of the SBIBD. For these matrices $\lambda(v - 1) = k (k - 1)$.

We note that for every SBIBD(v; k; λ) there is a complementary SBIBD(v; v − k; v − 2k + λ). One can be made from the other by interchanging the 0's of one with the 1's of the other. The usual use SBIBD convention that v > 2k and k > 2λ is followed.

Example 1: An SBIBD(7; 4; 2) = B and its complementary SBIBD(7; 3; 1) can be written with incidence matrices: they are still complementary if permutations of rows/ columns are applied to one and other permutations of rows/ columns to the other. This is because if P and Q are permutation matrices PBQ is equivalent to the SBIBD(7; 4; 2).

$$\text{SBIBD}(7;4;2) = \begin{bmatrix} 1\,1\,1\,0\,1\,0\,0 \\ 0\,1\,1\,1\,0\,1\,0 \\ 0\,0\,1\,1\,1\,0\,1 \\ 1\,0\,0\,1\,1\,1\,0 \\ 0\,1\,0\,0\,1\,1\,1 \\ 1\,0\,1\,0\,0\,1\,1 \\ 1\,1\,0\,1\,0\,0\,1 \end{bmatrix} \quad \text{SBIBD}(7;3;1) = \begin{bmatrix} 0\,1\,1\,0\,1\,0\,0 \\ 0\,0\,1\,1\,0\,1\,0 \\ 0\,0\,0\,1\,1\,0\,1 \\ 1\,0\,0\,0\,1\,1\,0 \\ 0\,1\,0\,0\,0\,1\,1 \\ 1\,0\,1\,0\,0\,0\,1 \\ 1\,1\,0\,1\,0\,0\,0 \end{bmatrix}$$

The second incidence matrix is still a complement of the incidence matrix of the first SBIBD even after permutations of its rows and/or columns have been performed.

In this work we will only use orthogonal to refer to matrices comprising real elements with modulus ≤ 1, where at least one entry in each row and column must be one. Hadamard matrices and weighing matrices are the best known of these matrices. We refer to [11, 9, 12] for definitions.

We now define our important concepts the orthogonality equation, the radius equation(s), the characteristic equation(s) and the weight of our matrices.

Definition 5. [Orthogonality equation, radius equation(s), characteristic equation(s), weight]

Consider the matrix $X = (x_{ij})$ comprising of the variables x and y. The matrix orthogonality equation

$XX^T = X^TX = \omega I_n$

yields two types of equations: the n equations which arise from taking the inner product of each row/ column with itself (which leads to the diagonal elements of ωI_n being ω) are called radius equation(s), $g(x; y) = \omega$, and the $n^2 - n$ equations, $f(x; y) = 0$, which arise from taking inner products of distinct rows of X (which leads to the zero o diagonal elements of ωI_n) are called characteristic equation(s). The orthogonality equation is $\sum_{j=1}^{n} x_{ij}^2 = \omega$. ω is called the weight of X.

Notation 2: We use X or X $(v; 2; \omega; \ldots)$ to denote a matrix of order v which has 2 variables (usually here x and y), which, when the variables are replaced by real numbers with modulus ≤ 1, (these may be negative numbers), the resultant matrix is orthogonal. The original 2-variable matrix and the resultant orthogonal matrix X are used to denote one-the-other.

Example 2: We consider the 2-variable X matrix given by:

$$X = \begin{bmatrix} x & y & y & y & y \\ y & x & y & y & y \\ y & y & x & y & y \\ y & y & y & x & y \\ y & y & y & y & x \end{bmatrix}$$

By definition, in order to become an orthogonal matrix, it must satisfy the radius and characteristic equations

$x^2 + 4y^2 = \omega;\ 2xy + 3y^2\ 0$

Thus we have, forcing x = 1, (since we require that at least one entry per row/ colmn is 1), $y = -\frac{2}{3}$ so $\omega = 3\frac{1}{3}$. The determinant is $\left(\frac{10}{3}\right)^{\frac{5}{2}}$ = 20.286. Hence we have an X = X(5;2; $\frac{10}{3}$; 20;5;20.286)

Mathematical Foundations for the 2-Variable Orthogonal Construction

Let X be a 2-variable matrix of order n: it will be written with variables x_1; x_2; ; x_s X = X(n; τ; ω; j_1; j_2;...; j_s; l_1; l_2;...; l_s; determinant) where n is the order,τ is the number of distinct variables (counting −x separately from x); j_1; j_2;...; j_s is the number of occurrences of the variables x_1; x_2;...; x_s if they occur the same number of times in each row and column, (however as this mostly does not happen these values are just omitted l_1; l_2;...; l_s is the total number of each variable in the whole matrix X, and the determinant. The original 2-variable matrix and the resultant orthogonal matrix X after the variables have been replaced by feasible entries/ values/ numbers are used to denote one-the-other.

Example 3: For SBIBD (7; 4; 2) consider circ (x; x; x; y; x; y; y) and for the complementary SBIBD (7; 3; 1), circ(y; x; x; y; x; y; y), or

$$\text{SBIBD (7; 4; 2)} = \begin{bmatrix} x\,x\,x\,y\,x\,y\,y \\ y\,x\,x\,x\,y\,x\,y \\ y\,y\,x\,x\,x\,y\,x \\ x\,y\,y\,x\,x\,x\,y \\ y\,x\,y\,y\,x\,x\,x \\ x\,y\,x\,y\,y\,x\,x \\ x\,x\,y\,x\,y\,y\,x \end{bmatrix} \quad \text{SBIBD (7; 3; 1)} = \begin{bmatrix} y\,x\,x\,y\,x\,y\,y \\ y\,y\,x\,x\,y\,x\,y \\ y\,y\,y\,x\,x\,y\,x \\ x\,y\,y\,y\,x\,x\,y \\ y\,x\,y\,y\,y\,x\,x \\ x\,y\,x\,y\,y\,y\,x \\ x\,x\,y\,x\,y\,y\,y \end{bmatrix}$$

Then considering the SBIBD (7; 4; 2): it has characteristic equation $2x^2 + 4xy + y^2 = 0$, and radius equation $\omega = 4x^2 + 3y^2$. Det(S) = $\omega^{\frac{7}{2}}$. The principal solution has

$x = 1$; $y = -2 + \sqrt{2}$; $\omega = 4x^2 + 3y^2 = 5.0294$.

Thus it gives a Cretan (7;2; 5.0294) matrix.

The SBIBD (7; 3; 1) with characteristic equation $x^2+4xy+2y^2 = 0$, and radius equation

$\omega = 4x^2 + 3y^2$, det(S) = $\omega^{\frac{7}{2}}$, (smaller than above values replacing y, has feasible solution

$x = 1$; $y = \frac{-2 + \sqrt{2}}{2}$; $\omega = 3x^2 + 4y^2 = 3.3431$.

Thus it gives a Cretan (7; 2; 3.3431) matrix.

The two determinants are 285:31 and 69:319 respectively.

Loosely we write 2-variable orthogonal SBIBD (7; 4; 2) and 2-variable orthogonal SBIBD (7; 3; 1) for the two matrices now given:

$$\begin{bmatrix} 1 & 1 & 1 & -2+\sqrt{2} & 1 & -2+\sqrt{2} & -2+\sqrt{2} \\ -2+\sqrt{2} & 1 & 1 & 1 & -2+\sqrt{2} & 1 & -2+\sqrt{2} \\ -2+\sqrt{2} & -2+\sqrt{2} & 1 & 1 & 1 & -2+\sqrt{2} & 1 \\ 1 & -2+\sqrt{2} & -2+\sqrt{2} & 1 & 1 & 1 & -2+\sqrt{2} \\ -2+\sqrt{2} & 1 & -2+\sqrt{2} & -2+\sqrt{2} & 1 & 1 & 1 \\ 1 & -2+\sqrt{2} & 1 & -2+\sqrt{2} & -2+\sqrt{2} & 1 & 1 \\ 1 & 1 & -2+\sqrt{2} & 1 & -2+\sqrt{2} & -2+\sqrt{2} & 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{-2+\sqrt{2}}{2} & 1 & 1 & \frac{-2+\sqrt{2}}{2} & 1 & \frac{-2+\sqrt{2}}{2} & \frac{-2+\sqrt{2}}{2} \\ \frac{-2+\sqrt{2}}{2} & \frac{-2+\sqrt{2}}{2} & 1 & 1 & \frac{-2+\sqrt{2}}{2} & 1 & \frac{-2+\sqrt{2}}{2} \\ \frac{-2+\sqrt{2}}{2} & \frac{-2+\sqrt{2}}{2} & \frac{-2+\sqrt{2}}{2} & 1 & 1 & \frac{-2+\sqrt{2}}{2} & 1 \\ 1 & \frac{-2+\sqrt{2}}{2} & \frac{-2+\sqrt{2}}{2} & \frac{-2+\sqrt{2}}{2} & 1 & 1 & \frac{-2+\sqrt{2}}{2} \\ \frac{-2+\sqrt{2}}{2} & 1 & \frac{-2+\sqrt{2}}{2} & \frac{-2+\sqrt{2}}{2} & \frac{-2+\sqrt{2}}{2} & 1 & 1 \\ 1 & \frac{-2+\sqrt{2}}{2} & 1 & \frac{-2+\sqrt{2}}{2} & \frac{-2+\sqrt{2}}{2} & \frac{-2+\sqrt{2}}{2} & 1 \\ 1 & 1 & \frac{-2+\sqrt{2}}{2} & 1 & \frac{-2+\sqrt{2}}{2} & \frac{-2+\sqrt{2}}{2} & \frac{-2+\sqrt{2}}{2} \end{bmatrix}$$

In all these Hadamard related cases ($v = 4t - 1$) (but not necessarily in all cases) the 2-variable orthogonal matrix with higher determinant comes from the SBIBD ($4t-1; 2t; t$)

While the SBIBD ($4t - 1; 2t - 1; t - 1$) gives a 2-variable orthogonal matrix with smaller determinant. These examples are given because they may give circulant SBIBD when other matrices do not necessarily do so.

CONSTRUCTIONS FOR 2-VARIABLE ORTHOGONAL MATRICES FROM SBIBD

We now use SBIBDs to construct 2-variable orthogonal matrices from SBIBD (x; y) s. We always, in m where, I_n is the $n \times n$ identity matrix, and a king 2-variable orthogonal SBIBD(x; y) from an SBIBD, change the ones of the SBIBD into x and the zeros of the SBIBD into y.

Theorem 1: Let X be made from an SBIBD ($v; k; \lambda$), B, by replacing the 1's with x and the 0's with y. Then X is a 2-variable orthogonal SBIBD (x; y) when X satisfies the matrix orthogonality equation.

$$XX^T = X^TX = \omega I_n \qquad (1)$$

Where, the radius equation gives

$$\omega = kx^2 + (v-k)y^2 \qquad (2)$$

A constant, ω, the weight of X, and I_v is the identity matrix of order v. These parameters also satisfy the characteristic equation:

$$\lambda x^2 + 2(k - \lambda)xy + (v - 2k + \lambda)y^2 = 0 \qquad (3)$$

The determinant is ${\omega_1}^{\frac{v}{2}}$.

The complementary design SBIBD ($v; v - k; v - 2k + \lambda$) satisfies equation (2) but with $\omega_2 = (v - k)x^2 + ky^2$. Its characteristic equation is

$$(v - 2k + \lambda)x^2 + 2(k - \lambda)xy + \lambda y^2 = 0: \qquad (4)$$

The determinant is ${\omega_2}^{\frac{v}{2}}$.

Let y_1, y_2 be the solutions for equation 3 and y_3, y_4 be the two solutions for equation 4. Then

Then, having forcing $x = 1$ we write the four solutions as

1. $y_1 = \frac{-(k-\lambda)+\sqrt{k-\lambda}}{v-2k+\lambda}$; $|y_1| \leq 1$ always;
2. $y_2 = \frac{-(k-\lambda)-\sqrt{k-\lambda}}{v-2k+\lambda}$

3. $y_3 = \frac{-(k-\lambda)+\sqrt{k-\lambda}}{\lambda}$

4. $y_4 = \frac{(k-\lambda)-\sqrt{k-\lambda}}{\lambda}$; $|y_4| \geq 1$ always;

So we have:

1. $X(v;2;\ k+(v-k)y^2;\ \frac{-(k-\lambda)+\sqrt{k-\lambda}}{v-2k+\lambda},\ 1;(k+(v-k)y^2)^{\frac{v}{2}})$

2. $X(v;2;\ k+(v-k)y^2;\ \frac{-(k-\lambda)-\sqrt{k-\lambda}}{v-2k+\lambda},\ 1;(k+(v-k)y^2)^{\frac{v}{2}})$ or

$X(v;2;\ (v-k)+ky^2;\ \frac{-(k-\lambda)+\sqrt{k-\lambda}}{v-2k+\lambda},\ 1;((v-k)+ky^2)^{\frac{v}{2}})$

Proof: We first consider the four potential solutions promised by the characteristic equations for the SBIBD (v; k; λ), and the complementary design SBIBD (v; v−k; v−2k + λ). We have called these four solutions y_1 and y_2, from the SBIBD (v; k; λ), and y_3 and y_4, from the SBIBD (v; v − k; v − 2k + λ).

When we pre-define x = 1 we are searching for matrices which require the value taken by the other variables should always have $|y_i| \leq 1$, that is $(y_i)^2 \leq 1$, to ensure entries are from the unit disk.

Simple multiplication and using $\lambda(v-1) = k(k-1)$ gives $y_1 \times y_4 = 1$ and $y_2 \times y_3 = 1$.

This means that $y_1 < 1 \Rightarrow y_4 > 1$ and $y_2 < 1 \Rightarrow y_3 > 1$ OR $y_2 > 1 \Rightarrow y_3 < 1$.

(However we must always be careful of the sign: for example $y_1 = \frac{-1}{3}$ and $y_4 = \frac{-2}{3}$ has $y_4 \leq y_1$ but $(y_1)^2 = \frac{1}{9} \leq (y_4)^2 = \frac{4}{9}$:) Now X, satisfying equations (2) and (3), and using $y_1 \leq 1$ give a two-variable orthogonal matrix by definition. The complementary design satisfies (2) and (4) and $y_4 \geq 1$ never gives a solution that has modulus ≤ 1.

Because $y_2 < 1 \Rightarrow y_3 > 1$ either the design or its complement must provide another solution that has modulus ≤ 1, we also have another 2-variable orthogonal SBIBD (x; y) by definition.

Example 4: We apply Theorem 1 to the SBIBD (45; 12; 3). We call the resulting 45×45 matrix X. X satisfies the orthogonality equation (1).

The usual design SBIBD (45; 12; 3), v ≡ 1 (mod 4), satisfies a radius equation and a characteristic equation, so we have

$\omega_1 = 12x^2 + 33y^2$ and $3x^2 + 18xy + 24y^2 = 0$:

This gives solutions $y_1 = \frac{-1}{2}$ and $y_2 = \frac{-1}{4}$ for x = 1 and y. We note that both these solutions have $|y| \leq 1$. So we have

$\omega_1 = 12 + 33 \times \frac{1}{4} = 20\frac{1}{4}$ and $\omega_2 = 12 + 33 \times \frac{1}{16} = 14\frac{1}{16}$

The determinants are 2.4806×10^{29} and 6.7828×10^{25} respectively. This means we have 2-level orthogonal X (45; 2) matrices:

$X(45;\ 2 : 20\frac{1}{4};\ 1;\frac{-1}{2})$ and $X(45;\ 2: 14\frac{1}{16};\ 1;\frac{-1}{4})$:

The complementary SBIBD (45; 33; 24) gives solutions -4 and -2 for x = 1 and y but no 2-variable orthogonal X (45; 2) matrices as here $|y| > 1$.

So the two solutions can both be found from the SBIBD (v; k;λ) for v; 2k; k < 2λ. We observe that the pair SBIBD and its complement still give two solutions.

Mathematics for Some Hadamard Matrix Related Constructions

There are three obvious Hadamard related constructions (but these are by no means all): those using SBIBD(4t − 1; 2t − 1; t − 1), those using the Menon difference sets and those using the twin prime difference sets. We illustrate using the first.

Corollary 1 [From Hadamard Matrices]: Suppose there exists an Hadamard matrix of order 4t, then there exists an SBIBD(4t−1; 2t−1; t−1). Hence for an X(v = 4t−1; 2), satisfying equations (1) and (3) with variables x and y, the solution x = 1, y = $\frac{-t+\sqrt{t}}{t}x$, $|y| = \frac{t-\sqrt{t}}{t} \le 1$, $\omega_1 = 2t+(2t-1)y^2$ and $\det(X) = (2t+(2t-1)y^2)^{(4t-1)/2}$

For X = X(v; 2), satisfying equations (1) and (4) with variables x and y, we have the Solution:

x = 1, y = $\frac{-t+\sqrt{t}}{t-1}x$, $|y| = \frac{t-\sqrt{t}}{t-1} \le 1$, $\omega_2 = 2t-1+ty^2$ and $\det(X) = (2t-1+ty^2)^{(4t-1)/2}$

CONCLUSION

We see from the La Jolla Repository of difference sets [10] that there exist (v; k;λ) difference sets and hence SBIBD(v; k;λ) for v = 4t + 1; 4t; 4t − 1; 4t − 2, where t is integer, which can be used to make circulant SBIBD(v; k; λ). We recall that the two Cretan(4t − 1; 2) where v ≡ 3 (mod 4) 2-level orthogonal matrices arise, one from the SBIBD and the other from its complement. This result is not necessarily so for v in other congruence classes.

The unexpected main conclusion is that the two Cretan(v; 2) we nd by this method can either:

1. both arise from the SBIBD(v; k;λ);
2. both arise from the SBIBD(v; v − k; v − 2k +λ);
3. one arises from the SBIBD(v; k; λ) and the other arises from the SBIBD(v; v−k; v − 2k + λ);

We conjecture that ω ≅ v will give unusual conditions.

N.A. Balonin has observed that the existence of some Cretan(4t−1; 2) 2-variable orthogonal matrices with the same parameters arise from the Cretan(4t−1; 2) – Singer difference set family and the Balonin-Sergeev (Mersenne family) [8], which are defined for orders 4t−1, where t is integer [7]. We have not considered the equivalence or other structural properties of Cretan matrices with the same parameters. All other useful references to this question may be found in [4].

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A Cornucopia of Research Questions

Tau-variable orthogonal matrices and Cretan matrices are a very new area of study. They have many research lines open: what is the minimum number of variables that can be used; what are the determinants that can be found for Cretan $(n; \tau)$ matrices; why do the congruence classes of the orders make such a difference to the proliferation of Cretan matrices for a given order; and the Cretan matrix with maximum and minimum determinant for a given order; can one be found with fewer levels?

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