CRETAN (4t + 1) MATRICES

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Purpose: We tried to obtain a Cretan(4t+1) matrix of order 4t+1, i.e. an orthogonal matrix whose elements have modulus \(\leq 1\). The only Cretan(4t+1) matrices previously published were of orders 5, 9, 13, 17 and 37. Results: In the paper, we give an infinite number of new Cretan(4t+1) matrices constructed by the use of regular Hadamard matrices, SBIBD(4t+1; k; \(\lambda\)), weighing matrices, generalized Hadamard matrices and Kronecker product. We introduce an inequality for the matrix radius and give a construction for a Cretan matrix of any order \(n \geq 5\). Practical relevance: Cretan(4t+1) matrices have direct practical applications to the problems of noise-immune coding, compression and masking of video information.

Keywords — Hadamard Matrices, Regular Hadamard Matrices, Orthogonal Matrices, Symmetric Balanced Incomplete Block Designs (SBIBD), Cretan Matrices, Weighing Matrices, Generalized Hadamard Matrices, 05B20.

Introduction

An application in image processing (compression, masking) led to the search for orthogonal matrices, all of whose elements have modulus \(\leq 1\) and which have maximal or high determinant.

Cretan matrices were first discussed, per se, during a conference in Crete in 2014. This paper follows closely the joint work of N. A. Balonin, Jennifer Seberry and M. B. Sergeev [1–3].

The orders 4t (Hadamard), 4t – 1 (Mersenne), 4t – 2 (Weighing) are discussed in [4–6]. This present work emphasizes the 4t + 1 (Fermat type) orders with real elements \(\leq 1\). Cretan matrices which are complex, based on the roots of unity or are just required to have at least one 1 are mentioned.

Preliminary Definitions

The absolute value of the determinant of any matrix is not altered by 1) interchanging any two rows, 2) interchanging any two columns, and/or 3) multiplying any row/or column by –1. These equivalence operations are called Hadamard equivalence operations. So the absolute value of the determinant of any matrix is not altered by the use of Hadamard equivalence operation.

Write \(I_n\) for the identity matrix of order \(n\), \(J\) for the matrix of all 1’s and let \(\omega\) be a constant. An orthogonal matrix, \(S\), of order \(n\), is square, has real entries and satisfies \(SS^T = nI_n\). The core of a matrix is formed by removing the first row and column.

A Cretan matrix, \(S\), of order \(n\) has entries with modulus \(\leq 1\) and at least one 1 per row and column. It satisfies \(SS^T = nI_n\) and so it is an orthogonal matrix. A Cretan(\(n; \tau; \omega\)) matrix, or CM(\(n; \tau; \omega\)) has \(\tau\) levels or values for its entries [1].

An Hadamard matrix of order \(n\) has entries \(\pm 1\) and satisfies \(HH^T = nI_n\) for \(n = 1, 2, 4t, t > 0\) an integer. Any Hadamard matrix can be put into normalized form, that is having the first row and column all plus 1s using Hadamard equivalence operations: that is it can be written with a core. A regular Hadamard matrix of order \(4m^2\) has \(2m^2 \pm m\) elements 1 and \(2m^2 \mp m\) elements –1 in each row and column (see [7, 8]).

Hadamard matrices and weighing matrices are well known orthogonal matrices. We refer to [2, 7–10] for more details and other definitions. The reader is pointed to [11–13] for details of generalized Hadamard matrices, Butson — Hadamard matrices and generalized weighing matrices.

For the purposes of this paper we will consider an SBIBD(\(v, k, \lambda\)), to be a \(n \times v\) matrix, with entries 0 and 1, \(k\) ones per row and column, and the inner product of distinct pairs of rows and/or columns \(\lambda\). This is called the incidence matrix of the SBIBD. For these matrices \(\lambda(v - 1) = k(k - 1)\), \(BB^T = (k - \lambda)I + \lambda J\) and \(\det B = k(k - \lambda)^{v-1}\).

For every SBIBD(\(v, k, \lambda\)) there is a complementary SBIBD(\(v, v - k, v - 2k + \lambda\)). One can be made from the other by interchanging the 0’s of one with the 1’s of the other. The usual SBIBD convention that \(v > 2k\) and \(k > 2\lambda\) is followed.

We now define our important concepts the orthogonality equation, the radius equation(s), the characteristic equation(s) and the weight of our matrices.

Definition 1 (orthogonality equation, radius equation(s), characteristic equation(s), weight). Consider the matrix \(S = (s_{ij})\) of order \(n\) comprising the variables \(x_1, x_2, ..., x_r\).

The matrix orthogonality equation

\[ S^T S = SS^T = \omega I_n \quad (1) \]
yields two types of equations: the \( n \) equations which arise from taking the inner product of each row/column with itself (which leads to the diagonal elements of \( \text{CM}_n \) being \( \omega \)) are called radius equation(s), \( g(x_1, x_2, \ldots, x_n) = \omega \), and the \( n^2 - n \) equations, \( f(x_1, x_2, \ldots, x_n) = 0 \), which arise from taking inner products of distinct rows of \( S \) (which leads to the zero off diagonal elements of \( \text{CM}_n \) are called characteristic equation(s)). Cretan matrices must satisfy the three equations: the orthogonality equation (1), the radius equation and the characteristic equation(s).

Notation: We use \( \text{CM}(n; \tau; \omega; \det) \) or just \( \text{CM}(n; \tau; \omega) \), where \( \tau_1, \tau_2, \ldots, \tau_t \) are the possible values (or levels) of the elements in CM.

**Inequalities**

Some inequalities are known for matrices which have real entries \( \leq 1 \). Hadamard matrices, \( H = (h_{ij}) \), which are orthogonal and with entries \( \pm 1 \) satisfy the equality of Hadamard’s inequality (2) [9]

\[
\det(HH^T) \leq \prod_{i=1}^{n} \sum_{j=1}^{n} |h_{ij}|^2, \tag{2}
\]

have determinant \( \leq n^2 \). Further Barba [14] showed that for matrices, \( B \), of order \( n \) whose entries are \( \pm 1 \):

\[
\det B \leq \sqrt{2n} (n-1)^{-\frac{n-1}{2}}
\]

or asymptotically \( \approx 0.858(n)^\frac{n}{2} \).

For \( n = 9 \) Barba’s inequality gives \( \sqrt{17 \times 8^4} = 16 888.24 \). The Hadamard inequality gives 19 683 for the bound on the determinant of the \( \pm 1 \) matrix of order 9. So the Barba bound is better for odd orders. We thank Professor Christos Koukouvinos for pointing out to us that the literature, see Ehlich and Zeller, [15], yields a \( \pm 1 \) matrix of order 9 with determinant 14 336. These bounds have not been met for \( n = 9 \).

Koukouvinos also pointed out that in Raghavaraao [16] a \( \pm 1 \) matrix of order 13 with determinant 14 929 920 \( \approx 1.49 \times 10^7 \) is given. This is the same value given for \( n = 13 \) given by Barba’s inequality. The Hadamard inequality gives 1.74 \( \times 10^7 \) for the bound on the determinant of the \( \pm 1 \) matrix of order 13.

These bounds have been significantly improved by Brent and Osborn [17] to give \( \leq (n+1)^\frac{n}{2} \).

Wojtas [18] showed that for matrices, \( B \), whose entries are \( \pm 1 \), of order \( n = 2 \pmod{4} \) we have

\[
\det B \leq 2(n-1)(n-2)^{\frac{n-2}{2}}
\]

or asymptotically \( \approx 0.736(n)^\frac{n}{2} \).

This gives a determinant bound \( \leq 73 \ 728 \) for order 10 whereas the weighing matrix of order 10 has determinant \( 9^5 = 59 \ 049 \).

We observe that the determinant of a \( \text{CM}(n; \tau; \omega; \det) \) is always \( \omega^2 \).

Hence we can rewrite the known inequalities of this subsection noting that only the Hadamard in equality applies generally for elements with modulus \( \leq 1 \). Thus we have:

**Theorem 1.** Hadamard — Cretan Inequality.

The radius of a Cretan matrix of order \( n \) is \( \leq n \).

**Two Trivial Cretan(\( n \)) Families**

The next two families are included for completeness.

**The Basic Family**

**Lemma 1.** Consider \( C = aI + b(J - I) \) of order \( n \), \( a, b \) variables. This gives a \( \text{CM} \left( n; \left( n; 2; 1 + \frac{4(n-1)}{(n-2)^2} \right) \right) \) of order \( n \), i.e., a \( \text{CM} \left( n; 2; 1 + \frac{4(n-1)}{(n-2)^2} \right) \).

**Proof.** Writing \( C \) with \( a \) on the diagonal and other elements \( b \), the radius and characteristic equations become

\[
a^2 + (n - 1)b^2 = \omega \text{ and } 2a + (n - 2)b = 0.
\]

Hence with \( a = 1 \) and \( b = -\frac{2}{n-2} \) we have \( \omega = 1 + \frac{4(n-1)}{(n-2)^2} \) for the required \( \text{CM}(n) \) matrix.

**Remark 1.** For \( n = 7, 9, 11, 13 \) this gives \( \omega = \frac{1}{25} \frac{1}{49} \frac{1}{81} \frac{1}{121} \) respectively. These determinants are very small. However they do give a \( \text{CM}(n; 2) \) for all integers \( n \)

**Known Families**

The following results may be found in [19] and [6].

**Proposition 1.** [Cretan(4\( t \))] There is a Cretan(4\( t \)): 2; 4\( t \) for every integer 4\( t \) for which there exists an Hadamard matrix.

**Proposition 2.** [Cretan(4\( t - 1 \))] There are Cretan(4\( t - 1 \); 2; \( \omega \); \( \omega = 4t + 1 + \sqrt{t} \) and \( \omega = \frac{2t^3 + t - 2t(2t - 1)^{\frac{1}{2}}}{(t - 1)^2} \) for every integer 4\( t \) for which there exists an Hadamard matrix.

The next two results are easy for the knowledgable reader and merely mentioned here.

**Proposition 3.** [Cretan(4\( t - 2 \))] There are Cretan(4\( t - 2 \); 3; \( k \)) whenever there is a W(4\( t - 2 \), \( k \))
SBIBD gives many inequivalent weighing matrix. For \( k = 4t - 3 \), the sum of two squares, and a \( W(4t - 2, 4t - 3) \) is known, the complex Cretan matrix \( CM(4t - 2; 3; 4t - 2) \) has elements \( i = \sqrt{-1}, 1 \) or \(-1\).

**Proposition 4.** [Cretan(pq)]. There are complex Cretan(pq; p; n), when ever there exists a generalized Hadamard matrix based on the \( pq \) th roots of unity.

The Additive Families

We will illustrate this construction using two Cretan matrices to give a Cretan matrix whose order is the sum of their orders. This shows how many possible matrices we might find for any \( n \) but again all the determinants are small.

**Lemma 2.** Let \( A \) and \( B \) be CM\((n_1; 3; \omega_1)\) and CM\((n_2; 3; \omega_2)\) respectively. Then \( A \oplus B \) given by

\[
\begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix}
\]

is a CM\((n_1 + n_2; 4; \omega)\) matrix of order \( n_1 + n_2 \) with \( \omega = \min(\omega_1, \omega_2) \). (Note it does not have one per row and column.)

**Remark 2.** We note using smaller CM\((n_1; \tau; \omega)\) gives many inequivalent CM\((n_1; \tau; \omega)\) for any order \( n = \sum n_i \) but the elements of all but the smallest sub matrix will not contribute 1 to the resulting Cretan matrix.

Now with \( n = n_1 + n_2 \) for \( 21 = 4 + 17, 5 + 16, 6 + 15, 7 + 14, 8 + 13, 9 + 12, 10 + 11 \) plus other combinations, the sub matrices of orders \( n_1 \) and \( n_2 \) contribute differently to \( \tau \) and \( \omega \). This means

**Proposition 5.** There is a Cretan\((n; \tau; \omega)\) for every integer \( n \).

In the section on Kronecker product of Cretan matrices we explore the same Proposition 5 for more interesting \( \tau \).

Constructions for Cretan\((4t + 1; \tau)\) Matrices

We now describe a number of constructions for Cretan\((4t + 1)\) matrices.

**Constructions using SBIBD**

- 2-level Cretan\((4t + 1)\) matrices via SBIBD\((v = 4t + 1, k, \lambda)\)

The following Theorem is a special case of the construction for 2-level Cretan\((v = 4t + 1)\) given in [6]. It also yields a valid CM\((37; 2)\).

**Theorem 2** [6]. Let \( S \) be a CM\((v = 4t + 1; 2; \omega; (a, b))\) based on SBIBD\((v = 4t + 1, k, \lambda)\) then \( a = 1 \), \( b = (k - \lambda) + \sqrt{k - \lambda} \) and \( \omega = k^2 + (v - k)b^2 \), provided \(|b| \leq 1\).

**Example 1.** Using the La Jolla Repository http://www.ccrwest.org/ds.html of difference sets we obtain an SBIBD\((37, 9, 2)\). Using Theorem 2 we obtain CM\((37; 2; 12.325; 1, 0.345)\) and CM\((37; 2; 9.485; 1, 0.132)\). The complementary SBIBD\((37, 28, 21)\) does not give any Cretan matrix as \(|b| > 1\).

We especially note the \((45, 12, 3)\) difference set, where the occurrence of the Cretan \( \begin{pmatrix} 45; 2; 20/14 \end{pmatrix} \) matrix and the Cretan \( \begin{pmatrix} 45; 2; 14 \end{pmatrix} \) matrices both arise from the SBIBD\((45, 12, 3)\); the complementary SBIBD\((45, 33, 24)\) does not yield any Cretan matrix.

**Example 2.** Orthogonal matrices of orders 13 and 21 may be constructed by using the SBIBD\((13, 4; 1)\) and SBIBD\((21, 5, 1)\) given in [20]. CM\( \begin{pmatrix} 13; 2; 9; 60; 1, \sqrt{3} \end{pmatrix} \) and CM\( \begin{pmatrix} 21; 2; 10; 1, -1/6 \end{pmatrix} \) are given in Fig. 1, a, b.

All the examples of SBIBD\((4t + 1, k, \lambda)\) that we have given from the La Jolla Repository have been constructed using difference sets. Most of those we give arise from Singer difference sets and finite geometries: these SBIBD\((p^{n+1} - 1)/(p - 1), (p^n - 1)/(p - 1), (p^{n+1} - 1)/(p - 1)\) difference sets are denoted as PG\((n, p)\). The bi-quadratic type constructions are due to Marshall Hall [21]. There are many SBIBD constructed without using difference sets.

- **Bordered Constructions**

We do not elaborate on the next theorem here but note it gives many Cretan matrices CM\((n; 1)\).

**Theorem 3.** The matrix \( C \) below can be used to construct many CM\((v + 1; \tau; \omega)\) with borders by replacing the matrix \( B \) by an SBIBD\((v, k, \lambda)\).

When a matrix \( C \) is written in the following form

\[
C = \begin{bmatrix}
x & s & \ldots & s \\
s & \ldots & \ldots & \ldots \\
\vdots & \ldots & \ddots & \ldots \\
s & \ldots & \ldots & 1
\end{bmatrix}
\]

\( B \) is said to be the core of \( C \) and the \( s \)'s are the borders of \( B \) in \( C \). \( C \) is said to be in bordered form. The variables \( x \) and \( s \) can be realized in the cases described below.

\[ a \rightarrow CM(13; 2; 9.60); b \rightarrow CM(21; 2; 10) \]
• Using Regular Hadamard Matrices

For constructions and constructions many of the known
Regular Hadamard Matrices the interested reader
is referred to [8, 7, 22].

Lemma 3. Let \( M \) be a regular Hadamard matrix
of order \( 4m^2 \) with \( 2m^2 + m \) positive elements per row
and column. Then forming \( C \) as follows

\[
C = \begin{bmatrix}
1 & s & \cdots & s \\
s & \cdots & \cdots & \cdots \\
\vdots & \cdots & 1 & M \\
s & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

gives a Cretan(\( 4n^2 + 1 ; 4; 1 \)) matrix or
CM \( (4m^2 + 1; 4; (0, 1, -\frac{1}{2m}, \frac{1}{2m}) \).

Proof. For \( C \) to be a Cretan matrix it must satisfy
the orthogonality, order and characteristic equations.

\[
CC^T = (1 + 4m^2 s^2)I_{4m^2 + 1} - (s^2 + 4m^2)I_{4m^2 + 1} = \alpha I_{4m^2 + 1}
\]

for the orthogonality equation, giving \( s = 0, \alpha = 1 \)
for the radius equation and 0 for the characteristic
equations.

Hence we have a matrix of order \( 4m^2 + 1 \) with
elements \( 0, 1, \pm \frac{1}{2m} \) — satisfying the required
Cretan equations.

Corollary 1. Since there exists a regular (sym-
metric) Hadamard matrix of order \( 4 = \pm 2^2 \),
\( 4^2 = 2^{2^2}, \ldots \), there is a Cretan = \( 2^{2^2} \cdots + 1; 4; 1 \) for \( 2 \)
a Fermat number.

Proof. Let \( S \) be the regular symmetric Hadamard
matrix of order 4. Then the Kronecker product

\[
S \times S \times \ldots \times S
\]
is the required core for the construction in Lemma 3.

Example 3. Purported examples of pure Fermat
matrices in Fig. 2, \( a, b \) for orders 5 and 17: levels \( a, b \)
are white and black colours, the border level \( s \) is
given in grey. However the reader is cautioned that
though the figures appear to be Cretan matrices
they are not. They are based on SBIBD, including
the regular Hadamard matrix \( SBIBD(4m^2, 2m \pm m, m \pm m) \) and require \( c = a \). We note though that when
\( c = a \times 1 \) the radius and characteristic equations do
not give meaningful real solutions.

Example 4. See Fig. 3, \( a, b \) for examples of a reg-
ular Hadamard matrix of order 36 and a purported
new Balonin — Seberry type of 3-level Cretan(37)
with complex entries that is an orthogonal matrix
of order 37. A real Cretan(37; 2) does exist from
Theorem 2 above (see example).

Using Normalized Weighing Matrix Cores

The next construction is not valid in the real
numbers. However we can allow Cretan matrices
to have complex elements and choose the diagonal to be \( i = \sqrt{-1} \).

Lemma 4. Suppose there exists a normalized
conference matrix, \( B \), of order \( 4t + 2 \), that is a
W(4t + 2, 4t + 1). Then \( B \) may be written as

\[
B = \begin{bmatrix}
i & 1 & \cdots & 1 \\
1 & \cdots & \cdots & \cdots \\
\vdots & \cdots & F & \cdots \\
1 & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

This is a Cretan matrix.

Removing the first row and column of \( B \) to study
the core \( F \) is unproductive.

Generalized Hadamard Matrices
and Generalized Weighing Matrices

We first note that the matrices we study here
have elements from groups, abelian and non-abeli-
an, (see [11–13, 23, 24] for more information) and
may be written in additive or multiplicative notation.
The matrices may have real elements, elements
\( 1, -1 \), elements \( |n| \leq 1 \), elements \( 1, i, i^2 = -1 \),
elements \( 1, i, -1, -i, i^2 = -1 \), integer elements \( a + ib, \)
\( i^2 = -1 \), \( n \)-th roots of unity, the quaternions \( 1 \) and \( i, j, k, \)
\( i^2 = j^2 = k^2 = -1, ijk = -1 \), \( (a + ib) + (c + id), \)
a, b, c, d, integer and \( i, j, k \) quaternions or otherwise
as specified.
We use the notations $B^T$ for the transpose of $G$, $B^H$ for the group transpose, $B^C$ for the complex conjugate of $B^T$, $B^Q$ for the quaternion conjugate and $B^V$ for the quaternion conjugate transpose.

In all of these matrices the inner product of distinct rows $a$ and $b$ is $a \cdot b$ or $ab^{-1}$ depending on whether the group is written in additive or multiplicative form.

- **Generalized orthogonality:** A generalized Hadamard matrix, or difference matrix, $GH(n, g)$, of order $h = gn$, over a group of order $g$, has the inner product of distinct rows the whole group $g$.

ORTHOGONALITY depends on the fact that the roots of unity add to zero.

- **Butson Hadamard matrix** [11]

$$B = \begin{pmatrix} 1 & 1 & 1 \\ a & b & ab \\ 1 & ab & a \end{pmatrix} ; \quad BB^C = 3I_3, \omega^3 = 1, 1 + \omega + \omega^2 = 0$$

is said to be a Butson Hadamard matrix. Orthogonality depends on the fact that the $n$th roots of unity add to zero.

- **A generalized Hadamard matrix** [11, 12, 13], $GH(np, G)$, where $G$ is a group of order $p$, can also be written in additive form for example:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

is a $GH(6, Z_3)$.

- **A generalized weighing matrix**, $W = GW(np, G, k)$ [23], where $G$ is a group of order $p$, has no nonzero elements in each column and $W$ is orthogonal over $G$. The following two matrices are additive and multiplicative $GW(5, Z_3)$, respectively:

$$\begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & 1 & 2 \\ 0 & 1 & * & 0 \\ 0 & 2 & 0 & * \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

* is zero but not the zero of the group.

**Theorem 4.** Any generalized Hadamard matrix or generalized weighing matrix is a $CM(n; \tau_1, \omega_1)$ and $CM(n_2; \tau_2; \omega_2)$ then the Kronecker product of $A$ and $B$ written $A \times B$ is a $CM(n_1n_2; \tau_1 \tau_2; \omega_1\omega_2)$ where $\tau$ depends on $\tau_1$ and $\tau_2$.

### Table 1. Some Cretan $CM(4t + 1), 3 \leq 4t + 1 < 199$

<table>
<thead>
<tr>
<th>From Regular Hadamard Matrices ($\omega = 1$)</th>
<th>$5$</th>
<th>$17$</th>
<th>$37$</th>
<th>$65$</th>
<th>$101$</th>
<th>$145$</th>
<th>$197$</th>
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<tbody>
<tr>
<td>From Difference Sets (ds)</td>
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<tr>
<td>$u$</td>
<td>$k$</td>
<td>$\lambda$</td>
<td>Existence</td>
<td>Difference set</td>
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<tr>
<td>13</td>
<td>4</td>
<td>1</td>
<td>All known</td>
<td>LG(2, 3)</td>
<td>Unique Hall [28]</td>
<td></td>
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</tr>
<tr>
<td>21</td>
<td>5</td>
<td>1</td>
<td>All known</td>
<td>LG(2, 4)</td>
<td>Unique Hall [28]</td>
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<tr>
<td>37</td>
<td>9</td>
<td>2</td>
<td>Exists</td>
<td>Biquadratic residue ds</td>
<td>Hall [28]</td>
<td></td>
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<tr>
<td>45</td>
<td>12</td>
<td>3</td>
<td>All known</td>
<td>—</td>
<td>La Jolla [20]</td>
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<tr>
<td>57</td>
<td>8</td>
<td>1</td>
<td>All known</td>
<td>LG(2, 7)</td>
<td>Unique Hall [28]</td>
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<td>73</td>
<td>9</td>
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<td>All known</td>
<td>LG(2, 8)</td>
<td>Unique Hall [28]</td>
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<td>85</td>
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<td>Biquadratic residue ds</td>
<td>Hall [28]</td>
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<td>133</td>
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<td>Biquadratic residue ds</td>
<td>Hall [28]</td>
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</tr>
</tbody>
</table>

Kronecker Product | All orders which are the product a known order and of prime power $= 3 \pmod{4}$
Example 5. From [6, 25] we see that CM(3; 2; 2.25), CM(7; 2; 5.03) and CM(7; 2; 3.34) exist so there exist CM(21; 3; 11.32) and CM(21; 3; 7.52). The Hadamard — Cretan bound gives, for $n = 21$, radius $\leq 21$.

From Balonin and Seberry [6] we have that since an $SBIBD\left( p^r,\frac{p^r-1}{2},\frac{p^r-3}{4}\right)$ exists for all prime powers $p^r \equiv 3 \pmod{4}$ there exist $CM(p^r; 2; \omega)$ for all these prime powers (see Proposition 2). Hence using Kronecker products in the previous theorem and writing $K$ as a product of prime powers we have.

**Theorem 5.** There exists a $CM(\tau; \omega; \omega) \omega > 1$ for all odd orders $n$, $n = \prod p^r p^\omega \ldots$, where $\rho$ is an order for which a Cretan $CM(\rho = 4t + 1)$ is known and $p^r p^\omega \ldots$ are any prime powers $=3 \pmod{4}$, for some $\tau$ and $\omega$.

Table 1 gives the integers for which $\rho$ is presently known. Similar theorems can be obtained for all even $n$.

**Remark 3.** We note that $\tau$ depends on the actual construction used. Combining $CM(n_1; 2; \omega_1 : (a, b))$ and $CM(n_2; 2; \omega_1 : (a, b))$ gives $CM(n_1 n_2; 3; \omega_12 : (a^2, ab, b^2))$. General formulae for $\tau$ from $CM$ with different levels are left as an exercise.

The Difference between Cretan($4t + 1; \tau$) Matrices and Fermat Matrices

The first few pure Fermat numbers are $v = 3$, 5, 17, 257, 65 537, 4 294 967 297,... We note these are all $=1 \pmod{4}$ and may be constructed using Corollary 1. Fig. 4 gives an early example of a Fermat matrix.

Finding 3-level orthogonal matrices of order $=1 \pmod{4}$ for non-pure Fermat numbers has proved challenging. Orders $n = 9$ and $n = 13$ are given in [4].

Orders $v = 2^{even} + 1$ called Fermat type matrices, pose an interesting class to study.

**Table 2.** Cretan 2-level and 3-level CM($4t \pm 1$), $3 \leq 4t + 1 \leq 199$

<table>
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Orders $4t + 1$, $t$ odd, are Cretan($4t + 1$) matrices; their order is neither a Fermat number ($2 + 1 = 3$, $2^2 + 1 = 4 + 1 = 5$, $2^{2^2} + 1 = 16 + 1 = 17$, $2^{2^3} + 1 = 256 + 1 = 257$, ...) nor a Fermat type number ($2^{even} + 1$). Examples of regular Hadamard matrices of order 36, giving the first CM($37; 3; 1$) matrix of order 37 [3] where 37 is not a Fermat number or Fermat type number, have been placed at [26]. They use regular Hadamard matrices as a core and have the same, as any other Hadamard matrix, level functions. We call them Cretan($4t + 1$) matrices and will consider them further in our future work.

Matrices of the Cretan($4t + 1$) family made from Singer difference sets (see [21]) also have orders belonging to the set of numbers $4t + 1$, $t$ odd: these are different from the three-level matrices of Balonin — Sergeev (Fermat) family [27, 19] with orders $4t + 1$, $t$ is 1 or even.

**Summary**

In this paper we have given new constructions for CM($4t + 1$). These are summarised in Table 1 for $4t + 1 < 200$. Table 2 gives 2-level and 3-level CM($4t \pm 1$).
Cretan matrices are a very new area of study. They have many research lines open: what is the minimum number of variables that can be used; what are the determinants and radii that can be found for Cretan \((n; \tau)\) matrices; why do the congruence classes of the orders make such a difference to the proliferation of Cretan matrices for a given order; find the Cretan matrix with maximum and minimum determinant for a given order; can one be found with fewer levels?

References


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We thank Professors Richard Brent, Christos Koukouvinos and Ilias Kotsireas for their valuable input to this paper. The authors also wish to sincerely thank Mr Max Norden, BBMgt(C.S.U.), for his work preparing the layout and LaTeX version of this article, and Mrs Tamara Balonin for her work preparing the text of the ‘word’ version. We acknowledge http://www.wolframalpha.com for the number calculations in this paper and http://www.mathscinet.ru for the graphics.

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References

Критические матрицы порядков 4 + 1

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Цель: дать критические матрицы Кетрэн (4t + 1) порядков 4 + 1 — ортогональные матрицы с элементами, ограниченными по модулю ≤1 (ранее опубликованы критические матрицы типа Кетрэн 4t + 1, определенные порядков 5, 9, 13, 17 и 37). Результаты: предложена ссылка на репозиторий SBIBD. Критические матрицы, ассоциированные с SBIBD, критические матрицы, обобщенные матрицы, ортогональные матрицы, симметричные матрицы, сбалансированный блочный дизайн (SBIBD), критические матрицы, взвешенные матрицы, обобщенные матрицы, обобщенные матрицы Адамара, 05B20.