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Generalized Mersenne Matrices and Balonin's Conjecture

A. M. Sergeev
St. Petersburg State University of Aerospace Instrumentation,
ul. Bol'shaya Morskaya 67, St. Petersburg, Russia 190000
e-mail: mbse@mail.ru

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Abstract—Properties of generalized Hadamard matrices and a conjecture on the existence of Mersenne matrices included into the former are discussed. A new classification of few-level quasi-orthogonal matrices, including matrices of even and odd orders, is presented. Matrices of Euler, Mersenne, and Hadamard are considered. The fundamental differences between the matrices of Mersenne and Fermat, which explain the failure of the proof of the Hadamard conjecture, are shown.

Keywords: information processing, compression, masking, orthogonal matrices, quasi-orthogonal matrices, Hadamard matrices, Belevitch matrices, Mersenne numbers, Fermat numbers, Euler–Fermat criterion

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1. INTRODUCTION

The wide use of digital technologies in the fields related to processing, storage, and transmission of images and video streams poses a question concerning the revision of the fundamentals on which the modern transformation of such information is based. For a great number of applied problems, Hadamard matrices obtained using different algorithms are employed.

The Hadamard conjecture on the multiplicity of orders of minimax quasi-orthogonal matrices with respect to four [1–3] lays the groundwork for the general theory of concordance between integer numbers and orders of orthogonal bases. The development of this theory is related to the extension of orthogonal bases by searching and studying quasi-orthogonal matrices of odd orders.

A Hadamard matrix is a square matrix $\mathbf{H}_n$ on the order of n that consists of the numbers {1, –1} and whose columns are orthogonal $\mathbf{H}_n^{\mathrm{T}}\mathbf{H}_n = n\mathbf{I}$, where $\mathbf{I}$ is the unit matrix. This matrix is minimax in the sense that, on the set of an orthonormal matrix of the same order, the maximum m (m-norm of the matrix) of absolute values of its elements is minimal [3]. Sylvester was the first to propose an algorithm for calculating minimax matrices on orders of $n = 2^k$ [1]. This one of the most widely used methods for calculating Hadamard matrices is based on the following formula:

$$\mathbf{H}_{2n} = \begin{pmatrix} \mathbf{H}_n & \mathbf{H}_n \\ \mathbf{H}_n & -\mathbf{H}_n \end{pmatrix}, \quad (1)$$

where $\mathbf{H}_1 = 1$ is used as the initial value.

Hadamard noted that Sylvester matrices have the extreme value of a determinant on the set of real matrices with elements that do not exceed 1. He found and published a pair of matrices $\mathbf{H}_{12}$ and $\mathbf{H}_{20}$ [2] that are not included into Sylvester's sequence of matrices but possess similar properties. Later, matrices and sequences of matrices were found that qualified as Hadamard matrices; this brought up the assumption that the necessary condition of their existence is a sufficient condition: the order of all (except for the first two matrices of Sylvester's sequence) Hadamard matrices is $n = 4k$.

Belevitch was the first to offer the quintessential generalization of Hadamard matrices [4]; he proposed similar matrices with elements {1, –1, 0} of orders that are multiple of two.

Odd numbers of sequences $4k + 1$ and $4k + 3$ (or $4k - 1$) were introduced by Fermat and Euler. Fermat found that any prime number of the form $4k + 1$ is representable in the form of a sum of two squared integer numbers uniquely. A prime number of the form $4k + 3$ cannot be represented in the form of a sum of squares. This fact is used in Belevitch's matrix theory with the Euler–Fermat criterion sending the expandability of the number $n-1$ into a sum of two squares used in checking the necessary conditions for their existence.

2. GENERALIZED HADAMARD MATRICES OF ODD ORDERS

In the conventional Hadamard matrix theory, the order is assumed to be even, so the generalization on odd orders is not possible. At the same time, odd orders appear in the theory of these matrices beginning with trivial existence theorems that are successfully proved. In the earlier Hadamard matrix theory, a so-called core on the order of $n-1$ appears. In particular, from the Gilman theorem follows that a Hadamard matrix on the order of n exists if $n-1$ is a prime number [5]. The Gilman theorem does not contain a converse proposition. For example, the Sylvester matrix on the order of $n = 16$ definitely exists, although $n-1 = 15$ is not a prime number. Thus, this and other theorems are concerned only with conditions for the existence of matrices of certain simple types but do not describe them as a whole.

In Belevitch's approach, the Hadamard matrices are proved compound matrices of doubled order. With the order of the generalized Belevitch matrix proved to be half as large as $n = 4k$, it would appear reasonable that the dichotomy can further be used when turning to odd orders. Thus, the Hadamard matrices have components of odd orders that, in turn, can be regarded as their generalizations. The first odd orders to be discussed in detail are related to the Mersenne numbers $n = 2^k - 1$.

Definition 1. A generalized Hadamard matrix of odd order is called a matrix with orthogonal columns $\mathbf{M}_n^{\mathrm{T}}\mathbf{M}_n = \omega(n)\mathbf{I}$, where $\mathbf{I}$ is the unit matrix, and $\omega(n)$ is the weight coefficient with two values of its elements $\{a, -b\}$ (levels).

Without losing generality, we assume that $a = 1$, $b \le a$. The Hadamard matrix satisfies this definition since $b = a$, $\omega(n) = n$ holds for it.

The number of levels of the generalized Hadamard matrix can exceed two (for example, three levels for the Belevitch matrix). We confine ourselves to considering the two-level matrices that are more close to Hadamard matrices.

Theorem 1. For odd orders of $n = 2^k - 1$, there exist generalized Hadamard matrices, which are hereinafter referred to as Hadamard–Mersenne matrices.

Proof. Let us consider the following modified Sylvester formula:

$$\mathbf{S}_{2n} = \begin{pmatrix} \mathbf{M}_n & \mathbf{M}_n \\ \mathbf{M}_n & \mathbf{M}_n^* \end{pmatrix}, \tag{2}$$

where $\mathbf{M}_n$ is a certain initial symmetric Hadamard–Mersenne matrix, and the matrix $\mathbf{M}_n^*$ is formed by rearranging the levels $a = 1$ and $-b$ (which generalizes the inversion of levels in Sylvester's algorithm). The matrix $\mathbf{S}_{2n}$ obtained using formula (2) is symmetrical and corresponds to the desired matrix in terms of the element composition, but its order is even and is one less than the order of the Mersenne matrix $\mathbf{M}_{2n+1}$. For the recursive matrix-to-matrix transition, the doubling of the order is not sufficient: the extra bordering of the matrix $\mathbf{S}_{2n}$ is required (addition of a column and a string).

Let us form the matrix $\mathbf{M}_{2n+1}$ using the following bordering of the matrix $\mathbf{S}_{2n}$ of form (2):

$$\mathbf{M}_{2n+1} = \begin{pmatrix} -\lambda & \mathbf{e}' \\ \mathbf{e} & \mathbf{S}_{2n} \end{pmatrix}, \tag{3}$$

where λ and $\mathbf{e}$ are the eigenvalue and eigenvector of the matrix $\mathbf{S}_{2n}$, respectively, that ensure the orthogonally of the matrix $\mathbf{M}_{2n+1}$, which immediately follows from the substitution of (3) into the orthogonally condition $\mathbf{M}_m^{\mathrm{T}}\mathbf{M}_m = \omega(m)\mathbf{I}$, where $m = 2n + 1$, and consideration of the obtained block inequalities. To initiate the recursive process of calculating the Hadamard–Mersenne matrices, the following matrix on the order of $n = 3$ with $b = a/2$ is proposed in [6]:

$$\mathbf{M}_3 = \begin{pmatrix} a & -b & a \\ -b & a & a \\ a & a & -b \end{pmatrix},$$

which corresponds to the first nontrivial Mersenne number. The obtained matrices will be symmetric two-level Hadamard–Mersenne matrices with orders of $n = 2^k - 1$.

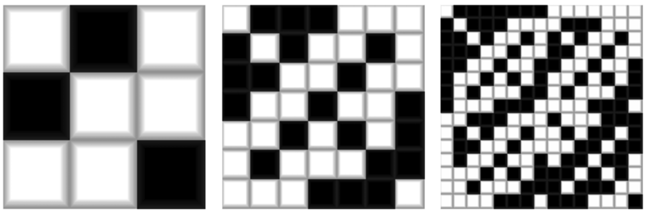

Fig. 1. Portraits of the Hadamard–Mersenne matrices $\mathbf{M}_3$, $\mathbf{M}_7$, and $\mathbf{M}_{15}$ on orders of Mersenne numbers.

The portraits of the first three matrices of the sequence are shown in Fig. 1. Hereinafter, the white square corresponds to unity, while the black square to the negative element b.

Obviously, they accompany Sylvester matrices but differ by odd orders from them. Less trivial is that, under conditions of the Gilman theorem [6], these matrices correspond to the order of $n-1$ and form the core of Hadamard matrices provided the rounding of the level modules up to integer numbers $a = b = 1$ [7]. Note that the rounding of the level values when transiting from Hadamard–Mersenne matrices to Hadamard matrices was first described in [7]. When transiting from Belevitch matrices to Hadamard matrices, the amplitude of the rounding is maximal and is 1 (for Mersenne matrices, $0 < b < a = 1$).

The advantage of Hadamard–Mersenne matrices is the analytically obtained characteristic equations for calculating their levels (table).

The level values and equations are given up to $n = 255$, but the table can easily be extended.

The Mersenne numbers belong to the subset of numbers of the form $4k-1$. Similar tables and matrices can be found for the orders described by the sequence of Fermat numbers $n = 2^{2^k} + 1$ that belong to the subset of numbers of the form $4k + 1$. Matrices that generalize Sylvester matrices with respect to the orders greater than $n = 2^k$ are discussed in [2]. Note that Hadamard matrices can be generalized by matrices for odd orders of $4k-1$ and $4k + 1$ with the case of Mersenne numbers being the most important due to their more frequent use as compared to Fermat numbers.

3. BALONIN'S CONJECTURE

Let us call attention to the fact that there are no matrices for odd orders of $n = 11$ and $n = 19$ in the series of generalized Hadamard–Mersenne matrices. Nevertheless, two two-level matrices $\mathbf{M}_{11}$ and $\mathbf{M}_{19}$ with orthogonal columns are found in [9] (see Figs. 2 and 3).

The feature of the matrices is that their negative levels are described by the general formula for a level module: $b = \frac{p \pm \sqrt{4p}}{p-4}a$, $p = n + 1$ (order of a neighboring Hadamard matrix). The negative sign in the

Equations and values of levels of **M** matrices

k	Matrix	Equation	Levels
1	$\mathbf{M}_1$	$b = a$	$b = a$
2	$\mathbf{M}_3$	$2b - a = 0$	$b = a/2$
3	$\mathbf{M}_7$	$b^2 - 4ab + 2a^2 = 0$	$b = (2 \pm \sqrt{2})a$
4	$\mathbf{M}_{15}$	$3b^2 - 8ab + 4a^2 = 0$	$b = 2a/3$ and $b = 2a$
5	$\mathbf{M}_{31}$	$7b^2 - 16ab + 8a^2 = 0$	$b = (8 \pm 2\sqrt{2})a/7$
6	$\mathbf{M}_{63}$	$15b^2 - 32ab + 16a^2 = 0$	$b = 4a/5$, $b = 4a/3$
7	$\mathbf{M}_{127}$	$31b^2 - 64ab + 32a^2 = 0$	$b = (32 \pm 4\sqrt{2})a/31$
8	$\mathbf{M}_{255}$	$63b^2 - 128ab + 64a^2 = 0$	$b = 8a/9$ and $b = 8a/7$

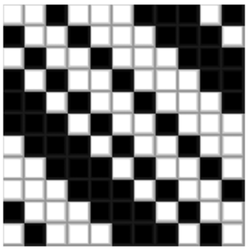

Fig. 2. Portrait of the matrix $\mathbf{M}_{11}$ (level $b = \frac{3-\sqrt{3}}{2}$).

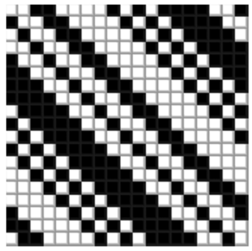

Fig. 3. Portrait of the matrix $\mathbf{M}_{19}$ (level $b = \frac{5-\sqrt{5}}{2}$).

numerator corresponds to a greater value of the determinant; for Hadamard matrices, the determinant is maximal on the set of the considered matrices.

It is easy to verify that the above formula describes any level of table or level of the Hadamard–Mersenne matrix that is not included into the table. Based on this fact and a great number of results obtained, an assumption is made in [9] that, for matrices of odd orders, a conjecture similar to the Hadamard conjecture can be formulated for the generalized Hadamard matrices.

For the exact formulation of the conjecture, the following definitions are required.

Definition 2.

A quasi-orthogonal matrix is called a square matrix $\mathbf{A}_n$ on the order of n with the reduced-to-unity maximums of the element modules for each column, which satisfies the square-law condition of the constraint $\mathbf{A}_n^{\mathrm{T}}\mathbf{A}_n = \omega(n)\mathbf{I}$, where $\mathbf{I}$ is the unit matrix and $\omega(n)$ is the weight of the matrix.

The weight $\omega = 1$ is characteristic of the orthogonal matrices that are not quasi-orthogonal matrices and, in particular, Hadamard matrices except for a trivial first-order matrix. At the same time, the matrices are rather close to the orthogonal ones obtained from $\mathbf{A}_n$ using the elementary normalization of their columns, whereupon the module-maximum element (m-norm) is reduced up to $m < 1$ for orders of $n > 1$. It can easily be seen that $|\det(\mathbf{A})| = \omega^{n/2}$ with $\omega = 1/m^2$. The Hadamard matrix $\mathbf{H}_n$, which possess the maximum determinant, has the minimum value $m = 1/\sqrt{n}$, i.e., is a particular case of quasi-orthogonal matrices with the weight $\omega = n$.

The weight is a characteristic that determines the matrix.

Definition 3.

The quasi-orthogonal Mersenne matrices $\mathbf{M}_n$ are called two-level matrices on orders of $n = 4k - 1$ with element values $\{1, -b\}$, where $b < 1$, which satisfy the square-law condition of the constraint $\mathbf{M}_n^{\mathrm{T}}\mathbf{M}_n = \omega(n)\mathbf{I}$; here, $\mathbf{I}$ is the unit matrix and $\omega(n) = \frac{(n+1)+(n-1)b^2}{2}$ is the changing weight with $b = 1/2$ for $n = 3$ and, in the other cases, $b = \frac{q+\sqrt{4q}}{q-4}$, $q = n + 1$ (order of the neighboring Hadamard matrices).

In this case, the generalized formula for the level b is used to calculate the weighting function and formulate the definition. Thus, Hadamard–Mersenne matrices and matrices from [9] fit this common definition. Such matrices are hereinafter referred to as Mersenne matrices, while the case of orders that is described by Mersenne numbers fits particular matrices that are found, in contrast to the more general case, using the modified (with respect to the fuzzy value of the order) Sylvester algorithm described in Theorem 1.

As to the other matrices, families of orders can be found whereby they can be constructed; all together, they are described by the following conjecture.

Balonin's conjecture. There exist Mersenne matrices on orders of $n = 4k - 1$.

In its sense, this assumption for quasi-orthogonal matrices of odd orders is an alternative to the Hadamard conjecture.

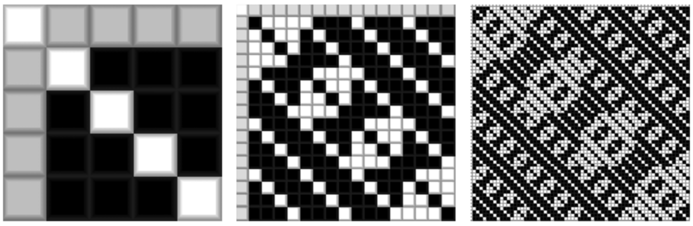

Fig. 4. Portraits of the Fermat matrices $\mathbf{F}_5$, $\mathbf{F}_{17}$, and $\mathbf{F}_{65}$.

4. CLASSIFICATION OF MINIMAX QUASI-ORTHOGONAL MATRICES

The theory of numbers and well-known number sequences are closely related to the theory of orthogonal bases, to construct which quasi-orthogonal matrices of Hadamard [2], Fermat [8], Mersenne [9], etc., are required.

There are only four variants of minimax quasi-orthogonal matrices with a small number of distinguishing elements—few-level **M** matrices [3]—depending on the remainder r on dividing the order n by four:

$r = 0$ are the Hadamard matrices (**H**) including the matrices of Sylvester's sequence;

$r = 1$ are the Fermat matrices (**F**) including the matrices on orders of Fermat numbers;

$r = 2$ are the Euler matrices (**E**) that substitute for the Belevitch matrices (**C**) that do not exist according to the Euler–Fermat criterion;

$r = 3$ are the Mersenne matrices (**M**) including the matrices on orders of Mersenne numbers.

Thus, the few-level **M** matrices involve the matrices **H**, **F**, **E**, and **M** from the set of quasi-orthogonal matrices wherein the sequences of Sylvester and Mersenne, by Balonin's conjecture, are the backbone.

The estimates of the density of covering the number axis by the values of matrix orders are based on the corresponding conjectures: the Hadamard conjecture (transfer of properties of matrices on Sylvester-sequence orders onto **H**) and the Balonin conjecture (transfer of properties of Hadamard–Mersenne matrices on Mersenne-sequence orders onto **M**).

Definition 4.

The quasi-orthogonal Fermat matrices $\mathbf{F}_n$ are called three-level matrices on orders of $n = 2^k + 1$ with elements $\{1, -b, s\}$, where $s \le b < 1$, which satisfy the square-law condition of the constraint $\mathbf{F}_n^{\mathrm{T}}\mathbf{F}_n = \omega(n)\,\mathbf{I}$, where $\mathbf{I}$ is the unit matrix and $\omega(n) = 1 + (n-1)s^2$ is the weight. The module levels are $b = s = 2/3$ for $n = 5$ (in the general case, $b = \frac{2n-p}{p}$), the canvas elements $s = \frac{\sqrt{nq - 2\sqrt{q}}}{p}$ constitute the first string and column except for the first unit element, $p = q + \sqrt{q}$, and $q = n - 1$ (order of the Hadamard matrix).

Theorem 2. On odd orders of $n = 2^k + 1$ (to which Fermat numbers belong), there exist Fermat matrices *for even numbers k*.

The proof of Theorem 2 can be constructed similarly to the proof of Theorem 1. The portraits of the first three Fermat matrices $\mathbf{F}_5$, $\mathbf{F}_{17}$, and $\mathbf{F}_{65}$ are shown in Fig. 4.

This type of approximating Hadamard matrices from the top is interesting for the theory of quasi-orthogonal matrices in that it is defined on the sequence of numbers $n = 4k + 1$ with gaps. Outside the basic (Sylvester) points, Fermat matrices are not found. This series of generalized Hadamard matrices is different. It is not related to the conjecture on the existence of matrices on orders of $n = 4k + 1$. Nevertheless, Fermat matrices are of great importance. If they always existed, it would be possible to interconnect the successive quadruples of matrices using the recurrence algorithm for increasing the matrix order by one (as is done in Theorem 1). Then, the conjectures of Balonin and Hadamard would become proved theorems. It is the gap (nonexistence of Fermat matrices) that is a formidable obstacle, which considerably hinders the proof of the statements. Let us supplement the description of quasi-orthogonal matrices with matrices on even orders of $n = 4k - 2$.

Definition 5.

The quasi-orthogonal Euler matrices $\mathbf{E}_n$ are called two-level matrices on orders of $n = 4k - 2$ with elements $\{1, -1, b, -b\}$, where $b < 1$, which satisfy the square-law condition of the constraint $\mathbf{E}_n^{\mathrm{T}}\mathbf{E}_n = \omega(n)\,\mathbf{I}$,

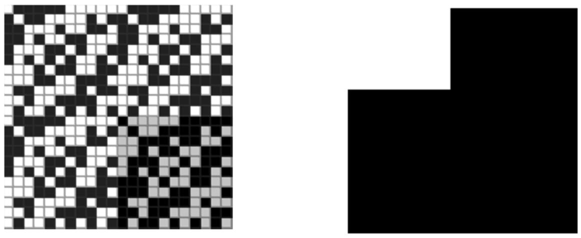

Fig. 5. Portrait of the matrix $\mathbf{E}_{22}$ and the histogram of modules of its elements.

where **I** is the unit matrix and $\omega(n) = \frac{(n+2)+(n-2)b^2}{2}$ is the changing weight of the matrix. The module level is $b = 1/2$ for $n = 6$ (in the general case, $b = \frac{q - \sqrt{8q}}{q-8}$) and $q = n + 2$ (order of the neighboring Hadamard matrices).

The definition and formulas for the weight and levels resulted from the possibility of calculating the quasi-orthogonal matrices of doubled (even) order using Sylvester's rule (1). Thus, there are no problems with the existence of these matrices. If Balonin's conjecture holds, then there exist Mersenne matrices and Euler matrices that are derivatives of the former [10]. They also exist for all the problem orders in Belevitch's matrix theory when $n - 1$ cannot be represented by the sum of two squared integers.

Figure 5 shows a portrait of the Euler matrix $\mathbf{E}_{22}$, which is constructed by quadrupling the Mersenne matrix $\mathbf{M}_{11}$, as well as a histogram of its element modules. Note that there is no Belevitch matrix on the order of $n = 22$. In this respect, we face more general matrices.

5. ADVANTAGES OF THE PROPOSED CLASSIFICATION OF M MATRICES

The Hadamard conjecture on the multiplicity of orders of quasi-orthogonal matrices with respect to four satisfies the well-known periodicity in the theory of numbers and rises a question concerning the other three types of matrices related to Hadamard matrices. When searching for the quasi-orthogonal matrices of odd orders [3, 6–10], which are close to Hadamard matrices in their properties, a class of few-level **M** matrices that are called generalized Hadamard–Mersenne matrices or (for simplicity) Mersenne matrices is distinguished. This is their essential difference from the well-known multilevel generalizations of Hadamard matrices [11] on odd orders, which are based on the feature of the maximum determinant.

The proposed classification covers all the possible cases of even and odd orders. In addition to Balonin's conjecture on the existence of Mersenne matrices on orders of $4k - 1$ (and even-order matrices generated by them), it is noted that there are no similar solutions on orders of $4k + 1$, which fill up the gaps in the series of Fermat matrices. Since the matrices **H**, **F**, **E**, and **M** allow one-to-one transformations into each other, such gaps may be responsible for the complexity of proving the Hadamard conjecture.

6. CONCLUSIONS

The conjectures on the existence of Hadamard matrices and Mersenne matrices are interrelated: these are similar assumptions for matrices of even and odd orders. The proof of one of them provides the validity for another [12]. This is a new conclusion that stems from the results of studying quasi-orthogonal matrices of odd orders inclusive.

These facts were unknown for Hadamard matrices inclusive, since the case of odd orders, due to the irrationality of the values of the matrix levels, had been studied insufficiently.

The proposed classification shows that the Mersenne matrices on orders of $4k - 1$ regulate the structure of matrices to the left and right of them, including the Hadamard and Euler matrices on even orders of $4k$ and $4k - 2$, as well as Fermat matrices on odd orders of $4k + 1$ if they exist. It is shown that Euler matrices can be a substitution for unavailable Belevitch matrices and generalize this type of matrices using a solution (which is less critical to the existence) of the problem of constructing the orthogonal basis.

In the absence of gaps among Fermat matrices [8], it would be possible to obtain all the symmetric orthogonal matrices on orders of number series (beginning with the initial $\mathbf{H}_1 = 1$) using the elementary increase of the bordering and recalculation of the negative elements $-b$ that tend to -1 with increasing n.

Gaps on orders of $4k + 1$ for Fermat matrices play a large part in the Hadamard matrix theory. Nevertheless, a further important question concerning the theory of few-level quasi-orthogonal matrices of odd orders is a question concerning the supplement of Fermat matrices with some other few-level matrices.

The regularities noted in the classification not only enhance the understanding of the complexity of the problem but also have independent implications for the application domain of coding, masking, and compression of information [13–16].

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