

## Research Article

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# Two-level Cretan matrices constructed using SBIBD

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**Abstract:** Two-level Cretan matrices are orthogonal matrices with two elements,  $x$  and  $y$ . At least one element per row and column is 1 and the other element has modulus  $\leq 1$ . These have been studied in the Russian literature for applications in image processing and compression. Cretan matrices have been found by both mathematical and computational methods but this paper concentrates on mathematical solutions for the first time.

We give, for the first time, families of Cretan matrices constructed using the incidence matrix of a symmetric balanced incomplete block design and Hadamard related difference sets.

**Keywords:** Hadamard matrices; orthogonal matrices; Cretan matrices; symmetric balanced incomplete block designs (SBIBD); difference sets

**MSC:** 05B20

## 1 Introduction

Families of *Cretan* matrices were first discussed, per se, during a conference in Crete in 2014 but had been previously studied in the Russian literature [3] using imprecise (or no) definitions and called, variously, Fermat, Hadamard, Mersenne, [1, 2, 4], Euler, M-matrices, C-matrices, and or conference families of matrices. We unify the previous definitions of families of Cretan matrices and give a construction which gives examples of all these matrix families.

Symmetric balanced incomplete block designs or  $(v, k, \lambda)$ -configurations or *SBIBD* $(v, k, \lambda)$  are of considerable use to statisticians undertaking medical or agricultural research. We use the usual *SBIBD* convention that  $v > 2k$  and  $k > 2\lambda$ .

We see from the La Jolla Repository of difference sets [6] that there exist  $(v, k, \lambda)$  difference sets for  $v = 4t + 1, 4t, 4t - 1, 4t - 2$  which can be used to make circulant *SBIBD* $(v, k, \lambda)$ .

In this and future papers we use some *names, definitions, notations* differently to how we they have been have in the past [2]. For example a Mersenne matrix (Russian) is in fact the core of a normalized Hadamard matrix (English). We do this to simplify the classification of Cretan matrices.

### 1.1 Definitions

**Definition 1.** A *orthogonal* matrix,  $S$ , has real entries and satisfies the *orthogonality equation*

$$S^T S = S S^T = \omega I_n, \quad (1)$$

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where  $I_n$  is the  $n \times n$  identity matrix, and  $\omega$ , the *weight*, is a constant real number.

**Definition 2.** A *Cretan matrix*,  $S$ , has entries with modulus  $\leq 1$ , and at least one 1 per row and column. It satisfies the *orthogonality equation*

$$S^T S = S S^T = \omega I_n$$

where  $I_n$  is the  $n \times n$  identity matrix, and  $\omega$ , the *weight*, is a constant real number.

**Definition 3.**  $S = (s_{ij})$  of order  $n$  will be called a  $\tau$ -*variable orthogonal matrix*, with variables  $x_1, x_2, \dots, x_\tau$  when it is orthogonal, satisfying  $S S^T = \omega I$  and for which  $\sum_{j=1}^n (s_{ij})^2 = \omega$ ,  $\omega$  a real constant, for all  $i$  and  $\sum_{i=1}^n s_{ji} s_{ki} = 0$  for each distinct pair of distinct rows  $j$  and  $k$ . A similar condition holds for the columns of  $S$ .  $S$  is a  $\tau$ -*level Cretan matrix* when the variables are replaced by suitable real numbers with modulus  $\leq 1$  where at least one variable is one. We write this as  $S = CM(n; \tau; \omega; \dots)$ .

In this paper we only study 2-variable orthogonal matrices of order  $n$  (or  $\nu$ ),  $S$  or  $S(n; 2; \omega; \dots)$ , written with the variables  $x$  and  $y$ . When the variables are replaced by real numbers with modulus  $\leq 1$ , (these may be negative numbers), the resultant matrix is orthogonal and called a Cretan matrix  $CM(n; 2; \omega; \dots)$ . The original 2-variable matrix and the resultant orthogonal matrix  $S$  are used to denote one-the-other.

We use the notations *Cretan*( $n$ ), *Cretan*( $n$ )-*SBIBD* and *Cretan-SBIBD* for Cretan matrices of 2-levels and order  $n$  constructed using *SBIBDs*.

We now define our important concepts: the *orthogonality equation*, the *radius equation(s)*, the *characteristic equation(s)* and the *weight* of our matrices.

**Definition 4 (Orthogonality equation, radius equation(s), characteristic equation(s), weight).** Consider the matrix  $S = (s_{ij})$ , of order  $n$ , comprising the variables  $x$  and  $y$ .

The *matrix orthogonality equation* is Equation 1, so  $S$  satisfies  $S^T S = S S^T = \omega = g(x, y) I_n$ . Equation 1 yields two types of equations: the  $n$  equations which arise from taking the inner product of each row/column with itself (which leads to the diagonal elements of  $\omega I_n$  being  $\omega$ ) and are called *radius equation(s)*,  $g(x, y) = \omega$ ; and the  $n^2 - n$  equations,  $f(x, y) = 0$ , which arise from taking inner products of distinct rows of  $S$  (which leads to the zero off diagonal elements of  $\omega I_n$ ) which are called *characteristic equation(s)*.

**Example 1.** We consider the 2-variable  $S$  matrix given by

$$S = \begin{bmatrix} x & y & y & y & y \\ y & x & y & y & y \\ y & y & x & y & y \\ y & y & y & x & y \\ y & y & y & y & x \end{bmatrix}.$$

By definition, in order to become an orthogonal matrix, it must satisfy the radius and characteristic equations

$$x^2 + 4y^2 = \omega, \quad 2xy + 3y^2 = 0.$$

To make a Cretan matrix we force  $x = 1$ , (since we require that at least one entry per row/column is 1), and the characteristic equation gives  $y = -\frac{2}{3}$ . Hence  $\omega = \frac{25}{9}$ . The determinant is  $(\frac{25}{9})^{\frac{5}{2}} = 12.86$ . This gives the Cretan(5; 2; ( $\frac{25}{9}$ ); 20; 5; 12.86) matrix  $S$ :  $\square$

## 2 Preliminary Definitions and Results for SBIBD

For the purposes of this paper we will consider an *SBIBD*( $\nu, k, \lambda$ ),  $B$ , to be a  $\nu \times \nu$  matrix, with entries 0 and 1,  $k$  ones per row and column, and the inner product of distinct pairs of rows and/or columns to be  $\lambda$ . This is called the *incidence matrix* of the *SBIBD*. For these matrices  $\lambda(\nu - 1) = k(k - 1)$ .

$$S = \begin{bmatrix} 1 & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & 1 & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & 1 & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & 1 & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & 1 \end{bmatrix} .$$

We note that for every  $SBIBD(v, k, \lambda)$  there is a complementary  $SBIBD(v, v - k, v - 2k + \lambda)$ . One can be made from the other by interchanging the 0's of one with the 1's of the other. The usual use  $SBIBD$  convention that  $v > 2k$  and  $k > 2\lambda$  is followed.

In this work we will only use orthogonal to refer to matrices comprising real elements with modulus  $\leq 1$ , where at least one entry in each row and column must be one. Hadamard matrices and weighing matrices are the best known of these matrices. We refer to [3, 7] for definitions.

The original 2-variable matrix and the resultant orthogonal matrix  $S$  after the variables have been replaced by feasible entries/values/numbers are used to denote one-the-other.

In all these Hadamard related cases ( $v = 4t - 1$ ) (but not necessarily in all cases) the 2-variable orthogonal matrix with higher determinant comes from the  $SBIBD(4t - 1, 2t, t)$  while the  $SBIBD(4t - 1, 2t - 1, t - 1)$  gives a 2-variable orthogonal matrix with smaller determinant. These examples are given because they may give circulant SBIBD when other matrices do not necessarily do so.

### 3 Orthogonal Matrices from SBIBD

We now use  $SBIBDs$  to construct 2-variable orthogonal matrices from  $SBIBD(x, y)$ s. We always, in making 2-variable orthogonal  $SBIBD(x, y)$  from an  $SBIBD$ , change the ones of the  $SBIBD$  into  $x$  and the zeros of the  $SBIBD$  into  $y$ .

We note there are two parts to the following theorem, one from the original design and the other from its complement. The solutions may appear the same but arise differently. Using the properties of an  $SBIBD(v, k, \lambda)$  design we have:

**Theorem 1.** [Two-level Cretan Matrices from SBIBD:I] *Let  $S$  be made from an  $SBIBD(v, k, \lambda)$ ,  $B$ , by replacing the 1's with  $x$  and the 0's with  $y$ . Then  $S$  is a Cretan( $v; 2; \omega_1$ ) or  $CM(v; 2; \omega_1)$  where*

$$\omega_1 = kx^2 + (v - k)y^2 \tag{2}$$

*is the radius. The characteristic equation is*

$$\lambda x^2 + 2(k - \lambda)xy + (v - 2k + \lambda)y^2 = 0 . \tag{3}$$

*The determinant is  $\omega_1^{\frac{v}{2}}$ .*

We see this leads to theoretical answers. Using the complementary  $SBIBD(v, v - k, v - 2k + \lambda)$  we get a similar result.

**Theorem 2.** [Two-level Cretan Matrices from SBIBD:II] *Let  $S$  be made from an  $SBIBD(v, v - k, v - 2k + \lambda)$ ,  $B$ , by replacing the 1's with  $x$  and the 0's with  $y$ . Then  $S$  is a Cretan( $v; 2; \omega_2$ ) or  $CM(v; 2; \omega_2)$  where*

$$\omega_2 = (v - k)x^2 + ky^2 \tag{4}$$

*is the radius. The characteristic equation is*

$$(v - 2k + \lambda)x^2 + 2(k - \lambda)xy + \lambda y^2 = 0 . \tag{5}$$

The determinant is  $\omega \frac{v}{2}$ .

We combine these as Corollary 1: *The Cretan 2-level Matrices from SBIBD Theorem.*

**Corollary 1.** [The Cretan 2-level Matrices from SBIBD Theorem] *Whenever there exists an SBIBD(v, k, λ) there exist two Cretan(order;τ; ω; y, x; determinant), or 2-level Cretan-SBIBD(v; 2), or S or CM, as follows,*

1. *one from the Cretan(v; 2; kx<sup>2</sup> + (v - k)y<sup>2</sup>;  $\frac{-(k-\lambda)-\sqrt{(k-\lambda)}}{v-2k+\lambda}$ , 1; determinant); made from the SBIBD(v, k, λ) and*
2. *one of the Cretan  $\left( v; 2; kx^2 + (v - k)y^2; \frac{-(k-\lambda)+\sqrt{(k-\lambda)}}{v-2k+\lambda}, 1; \text{determinant} \right)$ , made from the SBIBD(v, k, λ) or the Cretan  $\left( v; 2; (v - k)x^2 + ky^2; \frac{-(k-\lambda)-\sqrt{(k-\lambda)}}{\lambda}, 1; \text{determinant} \right)$ , made from the SBIBD(v, v - k, v - 2k + λ). □*

Note that the SBIBD(45, 12, 3) requires one set of options while SBIBD(43, 22, 11) requires the second set of options.

### 3.1 Hadamard Matrix Related Constructions

There are three obvious Hadamard related constructions (but these are by no means all): those using SBIBD(4t - 1, 2t - 1, t - 1), those using the Menon difference sets and those using the twin prime difference sets. We illustrate using the first. Balonin and Sergeev [4] gave results for 7 and 11.

**Corollary 2** (From Hadamard Matrices). *Suppose there exists an Hadamard matrix of order 4t, then there exists an SBIBD(4t - 1, 2t - 1, t - 1).*

*Hence for an S(v = 4t - 1; 2), satisfying Equations (2) and (3) variables we have a Cretan(4t - 1; 2; ω<sub>1</sub>; y, x; determinant) from the solution (that is y<sub>1</sub> or y<sub>2</sub>) for the 2-variable orthogonal matrix, by setting x = 1 in*

$$x, y_1 = \frac{-t + \sqrt{t}}{t}x, \omega_1 = 2t + (2t - 1)y^2 \text{ and } \det(S) = (2tx^2 + (2t - 1)y^2)^{\frac{4t-1}{2}}.$$

*For S = S(v; 2), satisfying Equations (4) and (5) we have a Cretan(4t-1;2;ω<sub>2</sub>;y,x; determinant) matrix from the solution (that is y<sub>3</sub> or y<sub>4</sub>) for the 2-variable orthogonal matrix, by setting x = 1. In particular for the SBIBD(4t - 1, 2t, t) we have*

$$1, y_3 = \frac{-t + \sqrt{t}}{t - 1}, \omega_2 = (2t - 1) + 2ty^2 \text{ and } \det(S) = (2t - 1 + 2ty^2)^{\frac{4t-1}{2}}.$$

*Explicitly for SBIBD(v, k, λ) = SBIBD(4t - 1, 2t - 1, t - 1) k = 2t - 1, λ we have the Cretan matrices:*

- *Cretan  $\left( 4t - 1; 2; 2t - 1 + 2ty_1^2; y_1 = \frac{t - \sqrt{t}}{t}, 1; (2t - 1 + 2ty_1^2)^{\frac{4t-1}{2}} \right)$ ;*
- *Cretan  $\left( 4t - 1; 2; 2t - 1 + 2ty_3^2; y_3 = \frac{-t + \sqrt{t}}{t - 1}, 1; 2t + (2t - 1)y_3^2 \right)^{\frac{4t-1}{2}}$ . □*

In all these Cretan-Hadamard cases (but not in all cases) the Balonin-Sergeev-Cretan(v) matrix with higher determinant comes from the SBIBD(4t - 1, 2t, t), t = 7 or 11, while the SBIBD(4t - 1, 2t - 1, t - 1) gives a Cretan(4t - 1) matrix with smaller determinant. These examples are given because they may give circulant SBIBD when other matrices do not necessarily do so.

To construct solutions of the second type, we choose the SBIBD, B, to be that with the smaller number of 1's per row and column, i.e., v > 2k. Similar results are obtained for all SBIBD constructed using difference sets, including the twin prime difference sets of Stanton and Sprott, and the Springer-SBIBD families which use hyperplane difference sets.

**Corollary 3** (Menon Sets and Regular Hadamard Matrices). *Suppose there exists a regular Hadamard matrix of order 4m<sup>2</sup>, then there exists an SBIBD(4m<sup>2</sup>, 2m<sup>2</sup> - m, m<sup>2</sup> - m). Hence we have two-level Cretan matrix, S, satisfying Equations (1) and (3) for y =  $\frac{m-1}{m+1}x$ . The principal solution, from the SBIBD(4m<sup>2</sup>, 2m<sup>2</sup> - m, m<sup>2</sup> - m) is well known as a regular Hadamard matrix with*

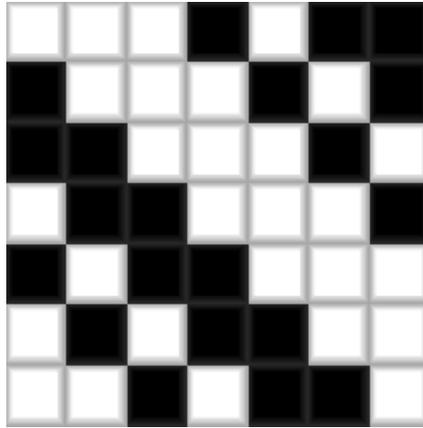


Figure 1: Orthogonal matrices for order 7: Balonin-Sergeev Family:  $CM(7,2,\omega)$   $CM(7,2,\omega)$

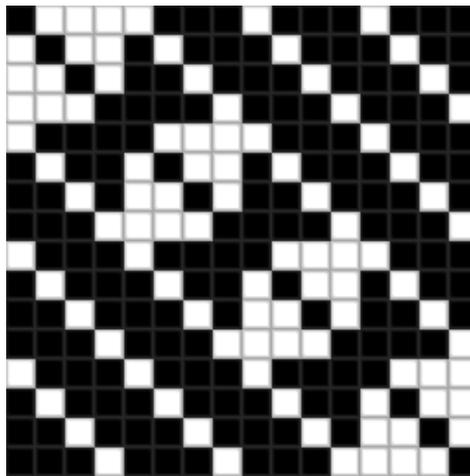


Figure 2: A regular Hadamard and 2-variable orthogonal matrix for order 16

$$x = 1, \quad y = 1, \quad \omega_1 = 4m^2, \quad \det(S) = (4m^2)^{2m^2}.$$

The second solution from the  $SBIBD(4m^2, 2m^2 + m, m^2 + m)$  gives

$$x = 1, \quad y = \frac{m-1}{m+1}x, \quad \omega_2 = \frac{4m^4}{(m+1)^2}, \quad \det(S) = \left(\frac{4m^4}{(m+1)^2}\right)^{2m^2}$$

for a two-level Cretan matrix with smaller determinant.  $\square$

**Example 2.** Consider  $p = 3$  which gives an  $SBIBD(13, 4, 1)$ . To generate this matrix we use  $\text{circ}(-b, a, -b, a, a, -b, -b, -b, a, -b, -b, -b, -b)$ , with characteristic equation  $a^2 - 6ab + 6b^2 = 0$ .

There are two solutions, see Fig. 1 where  $b = \frac{3+\sqrt{3}}{6}$ , and Fig. 1 where  $b = \frac{3-\sqrt{3}}{6}$ , so we have

$$a = 1, \quad b = \frac{1}{3 \mp \sqrt{3}} = \frac{3 \pm \sqrt{3}}{6}, \quad \omega = 4a^2 + 9b^2 = 7 \pm \frac{3\sqrt{3}}{2}, \quad = 9.5981 \text{ or } 4.4019$$

for the required two-level orthogonal matrices. They have  $\det(S) = \left(7 \pm \frac{3\sqrt{3}}{2}\right)^{\frac{13}{2}} = 2.4221 \times 10^6$  or  $1.5264 \times 10^4$ .  $\square$

## 4 Conclusions

We see from the La Jolla Repository of difference sets [6] that there exist  $(v, k, \lambda)$  cyclic difference sets and hence  $SBIBD(v, k, \lambda)$  for  $v = 4t + 1, 4t, 4t - 1, 4t - 2$ , where  $t$  is integer, which can be used to make circulant  $SBIBD(v, k, \lambda)$ . We recall that the two  $CM(4t - 1)$  where  $v \equiv 3 \pmod{4}$  2-level matrices arise, one from the  $SBIBD$  and the other from its complement. This result is *not necessarily* so for  $v$  in other congruence classes.

We note the existence of some  $CM(4t - 1)$  2-level matrices with the same parameters from the  $Cretan(4t - 1)$ -Singer family and the Balonin-Sergeev ( $CM(4t-1)$ -Mersenne family) [4], which are defined for orders  $4t - 1$ , where  $t$  is integer. We have not considered the equivalence or other structural properties of  $CM$  matrices with the same parameters. All other useful references to this question may be found in [3].

Matrices of the  $Cretan(4t + 1)$ -Singer family also have orders belonging to the set of numbers  $4t + 1, t$  odd: these are different from the three-level matrices of Balonin-Sergeev (Fermat) family [4] with orders  $4t + 1, t$  is 1 or even. The latter exist for orders  $v$ , where  $v - 1$  is order of a regular Hadamard matrix, described above via Menon sets.

Orders  $4t + 1, t$  is odd, are  $Cretan(4t + 1)$ - $SBIBD$  matrices; their order may be neither a Fermat number ( $2 + 1 = 3, 2^2 + 1 = 4 + 1, 2^{2^2} + 1 = 16 + 1, 2^{2^{2^2}} + 1 = 256 + 1, \dots$ ) nor a Fermat type number ( $2^{even} + 1$ ). The first  $CM(37; 3)$  uses regular Hadamard matrices as a core and has the same (as ordinary Hadamard matrices) level functions. We call them  $Cretan(4t + 1)$ - $SBIBD$  matrices and will consider them further in our future work.

The twin prime power difference sets allow us to have circulant  $Cretan(4t - 1)$ - $SBIBD$  in orders  $pq$  which were not previously known,  $pq = 35, 143, 323, \dots$ .

The main conclusion (about alternating matrices) follows: orders  $4k + 1, k$  odd, belong to alternating two- and three-level matrices of the Cretan-SBIBD-Singer and Balonin-Sergeev-Fermat families [4]. The sets of orders known for Cretan-SBIBD-Singer and Balonin-Sergeev-Mersenne families have orders in common, they (and hyper-plane SBIBD based matrices) are similar to each-other and distinct from the three level Balonin-Sergeev-Fermat family.

The unexpected main conclusion is that the two  $Cretan(v, k)$  we find by this method can either

1. both arise from the  $SBIBD(v, k, \lambda)$ ;
2. both arise from the  $SBIBD(v, v - k, v - 2k + \lambda)$ ;
3. one arises from the  $SBIBD(v, k, \lambda)$  and the other arises from the  $SBIBD(v, v - k, v - 2k + \lambda)$ ;

Cretan matrices are a very new area of study. They have a cornucopia of research lines open: what is the minimum number of variables that can be used; what are the determinants that can be found for  $Cretan(n; \tau)$  matrices; why do the congruence classes of the orders make such a difference to the proliferation of Cretan matrices for a given order; find the Cretan matrix with maximum and minimum determinant for a given order; can one be found with fewer levels? how can computational constructions help or be helped find optimal or near optimal solutions to problems?

We conjecture that  $\omega \cong v$  will give unusual conditions.  $\square$

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