COMBINATORIAL MATHEMATICS III, Lecture Notes in Mathematics, Vol. 452,
Springer-Verlag, Berlin-Heidelberg-New York, 1974, pp. 156-170.

ON THE MATRICES USED TO CONSTRUCT BAUMERT-HALL ARRAYS

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Four circulant (or type 1) (0,1,-1) matrices X_1, X_2, X_3, X_4 of order t with the property that each of the t^2 positions is non-zero in precisely one of the X_i and such that

$$X_1X_1^T + X_2X_2^T + X_3X_3^T + X_4X_4^T = tI_t$$

will be called *T-matrices*.

This paper studies the construction, use and properties of T-matrices giving a new construction for Hadamard matrices and some new equivalence results for Hadamard matrices and Baumert-Hall arrays.

1. INTRODUCTION

An *Hadamard matrix* $H = (h_{ij})$ is a square matrix of order n with elements $+1$ or -1 which satisfies the matrix equation

$$HH^T = H^TH = nI_n ,$$

where H^T denotes H transposed and I the identity matrix.

Unless specifically stated the order of matrices should be determined from the context. We use $-$ for -1 and J for the matrix with every element $+1$.

The matrices

$$[1] , \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix} , \begin{bmatrix} - & 1 & 1 & 1 \\ 1 & - & 1 & 1 \\ 1 & 1 & - & 1 \\ 1 & 1 & 1 & - \end{bmatrix} , \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{bmatrix}$$

are Hadamard matrices of order 1, 2, 4 and 4 respectively.

It can be shown (see [7] and [20]) that the order of an Hadamard matrix is necessarily 1, 2 or $4m$ for some $m = 1,2,3,\ldots$. It has

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been conjectured that Hadamard matrices of all these orders exist. For many years the first few unresolved cases have been 188, 236, 268 and 292. Richard J. Turyn has announced [1?] that he has found Hadamard matrices for the orders 188 and 236. Edward Spence has announced [10] the existence of the matrix of order 292. So the first few unresolved cases are now 268, 412 and 428.

The book [20] of Wallis, Street and Wallis gives all the constructions for Hadamard matrices known to one of us early in 1972 but many exciting results have been discovered more recently. For many elementary definitions we refer the reader to this book.

In 1944 Williamson [24] introduced a special type of Hadamard matrix

$$H = \begin{bmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{bmatrix}$$

based on the matrix representation of the quaternions.

Subsequently four (1,-1) matrices A, B, C, D of order m which satisfy

$$\text{(i)} \quad XY^T = YX^T , \qquad X,Y \in \{A,B,C,D\},$$

and $$\text{(ii)} \quad AA^T + BB^T + CC^T + DD^T = 4mI_m$$

have been called *Williamson matrices*. These matrices have been the subject of much recent study [8], [11], [12], [16], [17], [18], [22], [23], because

Theorem 1.1. *If there exist Williamson matrices of order* m *then there exists an Hadamard matrix of order* $4m$.

Definition 1.2. *The rows or columns of an array of indeterminates will be said to be* formally orthogonal *if realizing the indeterminates as any elements from a commutative ring causes the distinct rows or columns of the array to be pairwise orthogonal.*

Baumert and Hall (see [1]) in 1965 published a 12×12 array containing precisely 3 ±A's, 3 ±B's, 3 ±C's, 3 ±D's in each row and column. Furthermore the rows and columns were formally orthogonal. If the A, B, C, D are matrices which pairwise satisfy $XY^T = YX^T$ then

$$HH^T = I_{12} \times 3(AA^T + BB^T + CC^T + DD^T) .$$

More generally we consider

Definition 1.3. *A* $4t \times 4t$ *array of the indeterminates* $\pm A$, $\pm B$, $\pm C$, $\pm D$ *in which*

(i) each indeterminate, $\pm X$, *occurs precisely* t *times in each row and column, and*

(ii) the distinct rows are formally orthogonal

will be called a Baumert-Hall array.

Orthogonal designs which give an overview of Baumert-Hall arrays are studied in [3] and [4].

We have

Theorem 1.4. (Baumert-Hall). *If there exist a Baumert-Hall array of order* t *and four Williamson matrices of order* m *then there exists an Hadamard matrix of order* $4mt$.

Five years passed from the publication of the Baumert-Hall array of order 3 until Lloyd Welch [21] found his deceptively simple Baumert-Hall array of order 5.

Shortly after Welch's matrix was discovered Jennifer Wallis [14] and Richard J. Turyn [12] independently announced that a construction of Goethals and Seidel [5] was important in finding Baumert-Hall arrays. Their theorem is

Theorem 1.5. (Goethals-Seidel [5]) *If* X, Y, Z, W *are square circulant* $(1,-)$ *matrices of order* t, *if* $U = X - I$ *is skew-symmetric, and if*

$$XX^T + YY^T + ZZ^T + WW^T = 4tI_t ,$$

then

$$GS = \begin{bmatrix} X & YR & ZR & WR \\ -YR & X & -W^TR & Z^TR \\ -ZR & W^TR & X & -Y^TR \\ -WR & Z^TR & Y^TR & X \end{bmatrix} \qquad (*)$$

is a skew-Hadamard matrix of order $4t$ *when* $R = (r_{ij})$ *of order* t *is given by*

$$r_{ij} = \begin{cases} 1 , & j = t + 1 - i , \\ 0 & \text{otherwise} . \end{cases}$$

Wallis and Whiteman [15] showed how a similar matrix may be defined using an additive abelian group G.

Theorem 1.6. (Wallis-Whiteman [15]). *Let* X, Y, W *be type* 1 (1,-) *incidence matrices and* Z *be a type* 2 (1,-) *incidence matrix defined on the same abelian group of order* t *(see [19] for definitions). If*

$$XX^T + YY^T + ZZ^T + WW^T = 4tI_t$$

then

$$H = \begin{bmatrix} X & Y & Z & W \\ -Y^T & X^T & -W & Z \\ -Z & W^T & X & -Y^T \\ -W^T & -Z & Y & X^T \end{bmatrix} \qquad (\dagger)$$

is an Hadamard matrix of order $4t$. *Further if* $X - I$ *is skew,* H *is skew-Hadamard.*

The following array is an example of a Baumert-Hall array of order 3 constructed using the Goethals-Seidel method.

A	B	C	B	-C	D	C	D	-A	D	A	-B
C	A	B	-C	D	B	D	-A	C	A	-B	D
B	C	A	D	B	-C	-A	C	D	-B	D	A
-B	C	-D	A	B	C	-D	B	-A	C	-A	D
C	-D	-B	C	A	B	B	-A	-D	-A	D	C
-D	-B	C	B	C	A	-A	-D	B	D	C	-A
-C	-D	A	D	-B	A	A	B	C	-B	-D	C
-D	A	-C	-B	A	D	C	A	B	-D	C	-B
A	-C	-D	A	D	-B	B	C	A	C	-B	-D
-D	-A	B	-C	A	-D	B	D	-C	A	B	C
-A	B	-D	A	-D	-C	D	-C	B	C	A	B
B	-D	-A	-D	-C	A	-C	B	D	B	C	A

Definition 1.7. *Four type* 1 *(or circulant)* (0,1,-) *matrices* X_1, X_2, X_3, X_4 *of order* t *defined on the same abelian group (cyclic group)* G *of order* t *such that each of the* t^2 *positions is non-zero in precisely one of the* X_i *and satisfying*

$$X_1X_1^T + X_2X_2^T + X_3X_3^T + X_4X_4^T = tI_t$$

will be called T-*matrices.*

These matrices may be used to form Baume't-Hall arrays as follows:

Theorem 1.8. (Cooper-Wallis [2]). *Suppose there exist four T-matrices* X_1, X_2, X_3, X_4 *of order* t. *Further suppose that* A, B, C, D *satisfy* $MN^T = NM^T$ *and let*

$$\begin{aligned} X &= X_1 \times A + X_2 \times B + X_3 \times C + X_4 \times D , \\ Y &= X_1 \times -B + X_2 \times A + X_3 \times D + X_4 \times -C , \\ Z &= (X_1 \times -C + X_2 \times -D + X_3 \times A + X_4 \times B)R , \\ W &= X_1 \times -D + X_2 \times C + X_3 \times -B + X_4 \times A , \end{aligned}$$

with $R = (r_{ij})$ *defined on the elements of* G, $g_1, g_2, \ldots, g_t$ *by*

$$r_{\ell,j} = \begin{cases} 1 & \text{if } g_\ell + g_j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then (†) *gives a Baumert-Hall array of order* 4t.

T-matrices may be easily shown to have the following properties:

Lemma 1.9. *Let* X_1, X_2, X_3, X_4 *be* T-*matrices of order* t. *Further let* x_i *be the number of positive elements in each row and column of* X_i *and* w_i *the row sum (column sum) of each row (column) of* X_i. *Then*

(a) $2(x_1 + x_2 + x_3 + x_4) - (w_1 + w_2 + w_3 + w_4) = t$,

(b) $w_1^2 + w_2^2 + w_3^2 + w_4^2 = t$.

2. PREVIOUS CONSTRUCTION OF T-MATRICES

In [2] and [8] T-matrices are constructed, ad hoc, for various small orders using guessing and a little cyclotomy.

Apparently, Richard J. Turyn has used circulant T-matrices $X_1 = I$, X_2, X_3, $X_4 = 0$ of order $t = 2^a 10^b 26^c$, a,b,c non-negative integers, to get the first infinite class of T-matrices. We have been able to construct these matrices having (t-1)/2 non-zero entries in each row and column of X_2 and X_3. These matrices also exist for order 37.

3. A NEW CONSTRUCTION USING T-MATRICES

Definition 3.1. *Matrices (or linear combinations of matrices),* $A_i \times B_i$, *which may be used in the following array to form an Hadamard* *matrix will be called* Γ-matrices *(R has been defined in Theorem 1.8)*

$$\begin{bmatrix} A_1 \times B_1 & A_2R \times B_2 & A_3R \times B_3 & A_4R \times B_4 \\ -A_2R \times B_2 & A_1 \times B_1 & A_4^TR \times B_4 & -A_3^TR \times B_3 \\ -A_3R \times B_3 & -A_4^TR \times B_4 & A_1 \times B_1 & A_2^TR \times B_2 \\ -A_4R \times B_4 & A_3^TR \times B_3 & -A_2^TR \times B_2 & A_1 \times B_1 \end{bmatrix}$$

The A_i is called the *Goethals-Seidel part* (GS-part) and the B_i the *Williamson part* (W-part).

Theorem 3.2. *Let* $v \in \{1 + 2^a10^b26^c$, a,b,c non-negative integers$\}$ $\cup \{37\}$. *Suppose* $2v - 1$ *is a prime power* $\equiv 1 \pmod 4$; *then there exist* Γ-*matrices of order* $v(2v-1)$ *and an Hadamard matrix of Goethals-Seidel type of order* $4v(2v-1)$.

Proof. Let I, X_1, X_2, be the T-matrices of order v. Let X, Y be the type 1 (1,-1) incidence matrices of the $2 - \{2v-1;\ v-1;\ v-2\}$ supplementary difference sets described in [20 , p.283]. Then

$$X^T = X, \quad Y^T = Y, \quad XJ = YJ = -J, \quad XY^T = YX^T,$$

$$XX^T + YY^T = 4vI - 2J.$$

Now choose

$$A_1 \times B_1 = A = I \times J + (X_2+X_3) \times X,$$

$$A_2 \times B_2 = B = I \times J - (X_2+X_3) \times Y,$$

$$A_3 \times B_3 = C = I \times X - (X_2-X_3) \times Y,$$

$$A_4 \times B_4 = D = I \times Y + (X_2-X_3) \times X.$$

We write each of these in a form such as $A = A_1 \times B_1$ even though in fact A is a linear combination of such terms and use $A_4^T \times B_4$ for

$$R \times Y + (X_2-X_3)^TR \times X.$$

Now we may easily check that A, B, C, and D are the required Γ-matrices.

Example. Let $v = 3$. The $2 - \{5;\ 2;\ 1\}$ supplementary difference sets are $\{1,4\}$ and $\{2,3\}$ mod 5 and forming the circulant (1,-1) incidence matrices of these sets (where - is written for -1) we get

$$X = \begin{bmatrix} - & 1 & - & - & 1 \\ 1 & - & 1 & - & - \\ - & 1 & - & 1 & - \\ - & - & 1 & - & 1 \\ 1 & - & - & 1 & - \end{bmatrix} \text{ and } Y = \begin{bmatrix} - & - & 1 & 1 & - \\ - & - & - & 1 & 1 \\ 1 & - & - & - & 1 \\ 1 & 1 & - & - & - \\ - & 1 & 1 & - & - \end{bmatrix}.$$

Now for $v = 3$ the T-matrices are

$$I, \ X_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

and so

$$A_1 \times B_1 = \begin{bmatrix} J & X & X \\ X & J & X \\ X & X & J \end{bmatrix}, \quad A_2 \times B_2 = \begin{bmatrix} J & -Y & -Y \\ -Y & J & -Y \\ -Y & -Y & J \end{bmatrix},$$

$$A_3 \times B_3 = \begin{bmatrix} X & -Y & Y \\ Y & X & -Y \\ -Y & Y & X \end{bmatrix}, \quad A_4 \times B_4 = \begin{bmatrix} Y & X & -X \\ -X & Y & X \\ X & -X & Y \end{bmatrix}.$$

The required Hadamard matrix is

$$\begin{bmatrix}
J & X & X & J & -Y & -Y & X & -Y & Y & Y & X & -X \\
X & J & X & -Y & -Y & J & -Y & Y & X & X & -X & Y \\
X & X & J & -Y & J & -Y & Y & X & -Y & -X & Y & X \\
 & & & & & & & & & & & \\
-J & Y & Y & J & X & X & Y & -X & X & -X & -Y & Y \\
Y & Y & -J & X & J & X & -X & X & Y & -Y & Y & -X \\
Y & -J & Y & X & X & J & X & Y & -X & Y & -X & -Y \\
 & & & & & & & & & & & \\
-X & Y & -Y & -Y & X & -X & J & X & X & J & -Y & -Y \\
Y & -Y & -X & X & -X & -Y & X & J & X & -Y & -Y & J \\
-Y & -X & Y & -X & -Y & X & X & X & J & -Y & J & -Y \\
 & & & & & & & & & & & \\
-Y & -X & X & X & Y & -Y & -J & Y & Y & J & X & X \\
-X & X & -Y & Y & -Y & X & Y & Y & -J & X & J & X \\
X & -Y & -X & -Y & X & Y & Y & -J & Y & X & X & J
\end{bmatrix}$$

This construction does not give new Hadamard matrices as Williamson matrices are known for the orders given by the theorem.

4. CONSTRUCTION OF T-SETS FOR SMALL t

Definition 4.1. *The set of four first rows* T_1, T_2, T_3, T_4 *of four* T-*matrices* X_1, X_2, X_3, X_4 *will be called a* T-set.

To denote T-sets, the element j ($\bar{j}$) in T_i indicates that the jth term t_{ij} is +1 (-1); otherwise $t_{ij} = 0$. Only the non-empty sets T_i are listed, separated by slashes. Thus for $t = 5$, $12/3\bar{4}/5$ denotes the T-set

$$\{11000\}\ ,\ \{001\text{-}0\}\ ,\ \{00001\}\ ,\ \{00000\}\ .$$

Obviously, a T-set is transformed into a T-set by a *shift* ($j \to j + b$) or by a *multiplier* m relatively prime to t ($j \to mj$), where the arithmetic is mod t. (These are essentially operations on supplementary difference sets.)

Now to find all T-sets for a fixed t, we consider all possible representations

$$t = a_1{}^2 + a_2{}^2 + a_3{}^2 + a_4{}^2 = n_1 + n_2 + n_3 + n_4\ , \qquad (5)$$

where T_i has $a_i + b_i$ +1's, b_i -1's, $t - n_i$ 0's, and $n_i = a_i + 2b_i \geq a_i$. We search through all these possibilities to find T-sets.

The search is considerably simplified by the transformations noted above. For example, if $t = 7 = 2^2 + 1^2 + 1^2 + 1^2$, the only possibilities for the n_i are $\{4,1,1,1\}$ and $\{2,3,1,1\}$. Consider the latter case. We may choose the 3-set first. There are $\binom{7}{3} = 35$ of them, but by shifting we may assume that the 3-set contains the element 1, leaving only $\binom{6}{2} = 15$ possibilities. But all of these can be transformed to one of the sets $\{1,2,3\}$ and $\{1,2,4\}$. Thus there are essentially only two cases to consider.

In the table we give complete lists of T-inequivalent T-sets of orders $t = 3,5,7,9$. For each t we list the values of a_1, a_2, a_3, a_4, then n_1, n_2, n_3, n_4 in (5), then all T-inequivalent T-sets that occur with these parameters. [We note that most of the possible cases that do not in fact occur are ruled out by simple parity arguments. Only in a few cases was it necessary to test various assignments of +1 and -1 to rule out a case.]

Table of T-sets

t	a_1 a_2 a_3 a_4	n_1 n_2 n_3 n_4	T_1, T_2, T_3, T_4
3	1 1 1 0	1 1 1 0	1 / 2 / 3
5	2 1 0 0	2 1 2 0	1 2 / 3 $\bar{4}$ / 5
7	2 1 1 1	4 1 1 1	$\bar{1}$ 2 3 5 / 4 / 6 / 7
		2 3 1 1	1 2 $\bar{3}$ / 4 6 / 5 / 7
9	3 0 0 0	5 2 2 0	$\bar{1}$ 2 3 4 6 / 5 $\bar{7}$ / 8 $\bar{9}$
		3 2 2 2	1 2 4 / 3 $\bar{5}$ / 6 $\bar{9}$ / 7 $\bar{8}$
			1 2 4 / 3 $\bar{6}$ / 5 $\bar{7}$ / 8 $\bar{9}$
	2 2 1 0	4 4 1 0	1 2 3 $\bar{4}$ / 5 6 $\bar{7}$ 8 / 9
		4 2 3 0	1 2 3 $\bar{4}$ / 5 7 / 6 $\bar{8}$ 9
			1 2 $\bar{3}$ 4 / 5 7 / $\bar{6}$ 8 9
			$\bar{1}$ 2 3 5 / 6 8 / 4 $\bar{7}$ 9
			1 2 $\bar{3}$ 5 / 6 8 / $\bar{4}$ 7 9
		4 2 1 2	1 2 3 $\bar{4}$ / 5 8 / 6 $\bar{7}$ / 9
			1 2 3 $\bar{4}$ / 6 9 / 7 $\bar{8}$ / 5
			1 2 $\bar{3}$ 4 / 5 $\bar{8}$ / 6 7 / 9
			1 2 $\bar{3}$ 4 / 6 $\bar{9}$ / 7 8 / 5
			$\bar{1}$ 2 3 5 / 4 8 / 6 $\bar{9}$ / 7
			1 2 4 $\bar{5}$ / 3 7 / 6 $\bar{8}$ / 9
			1 2 4 $\bar{5}$ / 3 8 / 7 $\bar{9}$ / 6
			1 2 $\bar{4}$ 5 / 3 $\bar{7}$ / 6 8 / 9
			1 2 $\bar{4}$ 5 / 3 $\bar{8}$ / 7 9 / 6
		2 2 3 2	$\bar{1}$ 2 4 / 3 $\bar{5}$ / 6 9 / 7 8
			$\bar{1}$ 2 4 / 3 6 / 5 $\bar{7}$ / 8 9
			1 $\bar{2}$ 4 / 3 5 / 6 $\bar{9}$ / 7 8
			1 $\bar{2}$ 4 / 3 $\bar{6}$ / 5 7 / 8 9
			1 2 $\bar{4}$ / 3 5 / 6 9 / 7 $\bar{8}$
			1 2 $\bar{4}$ / 3 6 / 5 7 / 8 $\bar{9}$
		2 2 1 4	1 2 $\bar{3}$ $\bar{5}$ / 4 8 / 6 9 / 7

Note: for $t = 9$ we list the T-inequivalent 2-sets,...,7-sets.

2 : 12

3 : 123,124,147

4 : 1234,1235,1245,1247

5 : 12345,12346,12347,12457

6 : 123456,123457,124578

7 : 1234567

We also note the following remarkable occurrence. For each case with $t = 9$ we inserted the matrices X_1, X_2, X_3, X_4 in the Goethals-Seidel array to obtain a (0,1,-1) matrix M satisfying $MM^T = 9I_{36}$. The Smith Normal form (see [9,p.44]) was then calculated: necessarily there are r 1's, $(36-2r)$ 3's, and r 9's. For each case corresponding to the decomposition $9 = 3^2$, we found $r = 16$. For each case corresponding to $9 = 2^2 + 2^2 + 1^2$, we found $r = 18$.

5. SOME THOUGHTS ON EQUIVALENCE

Definition 5.1. *Two* (0,1,-1) *matrices* X *and* Y *of order* t *will be called* T-equivalent *if their first rows* $X' = \{x_i\}$ *and* $Y' = \{y_i\}$, $i = 1,\ldots,t$, *given in the notation used above, can be obtained from one another by using multipliers and shifts, i.e. if* $x_i = my_i + b \pmod t$, *for integers* m, b *and* $i = 1,\ldots,t$.

Definition 5.2. *Two matrices* A *and* B, *of order* n *will be called* H-equivalent *or* Hadamard equivalent *if there exist* (0,1,-1) *matrices* P *and* Q *with* $|\det P| = |\det Q| = 1$ *such that*

$$B = PAQ.$$

First we show that alteration of T-sets by shifting does not alter Hadamard equivalence.

Let $T = (t_{ij})$ of order n be defined by

$$t_{12} = t_{23} = \ldots = t_{n-1,n} = t_{n,1} = 1$$

and all other elements zero. Let A, B, C, D be polynomials in T and $S = T^x$ for some integer x. Let $R = (r_{ij})$ be the matrix of order n with

$$r_{1,n} = r_{2,n-1} = \ldots = r_{n-1,2} = r_{n,1} = 1$$

and all other elements zero.

Then A, B, C, D, S and their transposes pairwise commute, $S^TR = RS$, $R^T = R$ and $SS^T = I$.

Let

$$P = \begin{bmatrix} A & BR & CR & DR \\ -BR & A & D^TR & -C^TR \\ -CR & -D^TR & A & B^TR \\ -DR & C^TR & -B^TR & A \end{bmatrix}$$

and

$$Q = \begin{bmatrix} AS & BSR & CSR & DSR \\ -BSR & AS & (DS)^TR & -(CS)^TR \\ -CSR & -(DS)^TR & AS & (BS)^TR \\ -DSR & (CS)^TR & -(BS)^TR & AS \end{bmatrix}.$$

Then

$$P = \begin{bmatrix} (S^T)^2 & & & \\ & I & & \\ & & I & \\ & & & I \end{bmatrix} Q \begin{bmatrix} S & & & \\ & S^T & & \\ & & S^T & \\ & & & S^T \end{bmatrix}$$

and so P and Q are Hadamard equivalent.

Also

$$\begin{bmatrix} -I & & & \\ & -I & & \\ & & & -R \\ & & R & \end{bmatrix} P \begin{bmatrix} & R & & \\ -R & & & \\ & & I & \\ & & & I \end{bmatrix}$$

$$= \begin{bmatrix} -B & AR & CR & DR \\ -AR & -B & D^TR & -C^TR \\ -CR & -D^TR & -B & A^TR \\ -DR & C^TR & -A^TR & -B \end{bmatrix}$$

which is the same as P with A and B interchanged except that the interchanging of A and B has forced a change in sign of B. Also

$$\begin{bmatrix} I & & & \\ & -I & & \\ & & R & \\ & & & -R \end{bmatrix} P \begin{bmatrix} & -R & & \\ -R & & & \\ & & & -I \\ & & -I & \end{bmatrix}$$

$$= \begin{bmatrix} B & AR & DR & CR \\ -AR & B & C^TR & -D^TR \\ -DR & -C^TR & B & A^TR \\ -CR & D^TR & -A^TR & B \end{bmatrix},$$

which resembles P except that the interchanging of A and B has forced C and D to interchange too.

6. UNANSWERED QUESTIONS ON EQUIVALENCE

1. We noted for T-matrices of order 9 the decompositions into squares $9 = 3^2 + 0^2 + 0^2 + 0^2$ and $9 = 2^2 + 2^2 + 1^2 + 0^2$ gave T-inequivalent T-matrices and Z-inequivalent (P, Q of definition 5.2 have integer entries) and hence H-inequivalent weighing matrices W(36,9). Do different decompositions into squares always give Z-inequivalent and hence H-inequivalent (i) weighing matrices (ii) Hadamard matrices (iii) Baumert-Hall arrays?

2. If the X and Y of (*) are interchanged is the new Hadamard matrix or Baumert-Hall array H-equivalent to the old one? In section 5 it was observed that interchanging X and Y of (*) either (i) induced Z and W to interchange or (ii) Y to be replaced by -Y. Does this interchanging lead to H-inequivalence?

3. Let X_1, X_2, X_3, X_4 and Y_1, Y_2, Y_3, Y_4 be two sets of T-matrices of order n corresponding to the same decomposition of n into squares. Further suppose X_i is not T-equivalent to Y_j for any $i, j \in \{1,2,3,4\}$. Prove the Hadamard matrices and Baumert-Hall arrays formed from the X_i and Y_i, $i \in \{1,2,3,4\}$ are H-inequivalent.

4. Let X_1, X_2, X_3, X_4 be T-matrices of order n. Does the use of

$$
\begin{aligned}
X &= X_1 \times A + X_2 \times B + X_3 \times C + X_4 \times D \\
Y &= X_1 \times -B + X_2 \times A + X_3 \times -D + X_4 \times C \\
Z &= (X_1 \times -C + X_2 \times D + X_3 \times A + X_4 \times -B)R \\
W &= X_1 \times -D + X_2 \times -C + X_3 \times B + X_4 \times A
\end{aligned}
$$

in Theorem 1.7 instead of the X, Y, Z, W given in the enunciation there lead to H-inequivalent Baumert-Hall arrays or Hadamard matrices?

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