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A note on the singular-value decomposition of circulant Jacket matrices

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Abstract— In this paper a fast algorithm for the singular-value based decomposition motivated by circulant Jacket matrices is presented. The present decomposition, which is an extension of the singular-value-based decomposition of circulant matrices in essence, is a contribution to a problem of searching for a family of Jacket matrices of large size. Since the proposed circulant Jacket matrix is orthogonal, this algorithm is available in signal processing, image compression and communications.

***Index Terms*— Jacke, Toeplitz and circulant matrices; singular-value decomposition**

1. Introduction

The matrices discussed in this paper, such as Toeplitz [3], circulant [1] and [3-4], Hadamard [5] and Jacket [7] has an important role in numerical analysis and signal processing. The high practical value of Hadamard transformation such as center weighted Hadamard transformation is when it represents signal and images. The class of Jacket matrices also contains the class of real and complex Hadamard matrices. A special kind of Toeplitz matrix is the so called circulant matrix. Jacket matrices that are also Toeplitz matrices are in fact circulant Jacket (CJ) matrices. In this paper we present a very simple decomposition of CJ matrices which can provide an excellent eigenvalue and singular-value decomposition strategies. Eigenvalue and singular-value decomposition methods are intensively used in signal processing for solving numerous problems, whereas methods based on matrices with spontaneously computable inverse matrices are broadly used in practice. The CJ matrices also possess this excellent property. The Jacket matrices are a generalization of complex Hadamard matrices. A square $n \times n$ matrix $M = (m_{ij})$ is called *Jacket matrix* [6] if its inverse satisfies

$$\{M^{-1}\}_{i,j} = \frac{1}{nm_{j,i}}, \sum_{k=1}^{n} \frac{m_{i,k}}{m_{j,k}} = n\delta_{i,j}, i, j = 1, \ldots, n, \tag{1}$$

where $\delta_{i,j}$ is the Kronecker delta - a function of two variables, usually integers,

$$\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}. \tag{2}$$

If C is nonzero constant then the definition of Jacket matrix can be rewritten as follows. A square $m \times m$ matrix

$$J_m = \begin{pmatrix} j_{0,0} & j_{0,1} & \cdots & j_{0,m-1} \\ j_{1,0} & j_{1,1} & \cdots & j_{1,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ j_{m-1,0} & j_{m-11} & \cdots & j_{m-1,m-1} \end{pmatrix}$$

is called Jacket matrix if its normalized element-inverse transposed

$$J_m^{\star} = \frac{1}{C} \begin{pmatrix} 1/j_{0,0} & 1/j_{0,1} & \cdots & 1/j_{0,m-1} \\ 1/j_{1,0} & 1/j_{1,1} & \cdots & 1/j_{1,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1/j_{m-1,0} & 1/j_{m-11} & \cdots & 1/j_{m-1,m-1} \end{pmatrix}^T$$

satisfies $J_m J_m^{\star} = I_m$. A Jacket matrix in which all entries are of modulus 1 is called a complex *Hadamard matrix* [5]. It is easy to see that if K is a jacket matrix, then for every permutation matrices P_1, P_2 and for every invertible diagonal matrix D_1, D_2 the matrix $H = P_1 D_1 K D_2 P_2$ is a jacket matrix as well. Jacket matrices related in this fashion are called equivalent *equivalent*. Finding all jacket matrices up to equivalence turns out to be a challenging problem, and has been solved only up to orders $n \leq 5$ [2].

The property of Jacket matrices is, that for any 2 different rows $(a_{i,1}, \ldots, a_{i,n}$ and $(a_{j,1}, \ldots, a_{j,n})$ it is necessary to have the Jacket condition

$$\sum_{s=1}^{n} \frac{a_{i,s}}{a_{j,s}} = 0. \tag{3}$$

Requirement of (1) instead of usual inner product give as bad algebraic properties, about Jacket matrices – for example multiplication of two Jacket matrices in general is not Jacket matrix. However, Jacket matrices have some interesting combinatorial properties. For example if we multiply by non-zero element some row or column, then the matrix remain Jacket. This type of equivalence operation split the space in large classes of matrices. In orthogonal case we can multiply only by ± 1.
We conclude this section with some examples of Jacket matrices.

Examples of Jacket matrices are the following matrices of order $n = 2$ and 3, respectively: $A_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$,

where $\omega^2 + \omega + 1 = 0$, i.e. $\omega = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} \in \mathbb{C}$, so alternative notation is $A_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} - \frac{\sqrt{3}}{2} & -\frac{1}{2} + \frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} + \frac{\sqrt{3}}{2} & -\frac{1}{2} - \frac{\sqrt{3}}{2} \end{pmatrix}$. Numerous examples can be found in [7]. Some of the Jacket matrices can have parameters. For example for every nonzero a the matrix $A_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -a & a & -1 \\ 1 & a & -a & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$ is not only Jacket but also symmetric. Also, it is easy to check that the Vandermonde matrix of n-th roots [10] of unity $x_1, \ldots, x_n$, $W == \begin{pmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_n \\ x_1^2 & x_2^2 & \ldots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \ldots & x_n^{n-1} \end{pmatrix}$ is a Jacket matrix, too.

2. Toeplitz and circulant matrices

A *Toeplitz matrix* is a $n \times n$ matrix $T_n = \{t_{k,j}; k, j = 0,1,\ldots,n-1\}$, where $t_{k,j} = t_{k-j}$ for some elements $t_{-n+1}, \ldots, t_{-1}, t_0, t_1, \ldots, t_{n-1} \in \mathbb{C}$. In other words the Toeplitz matrixes have diagonal-constant elements and have the form

$$T_n = \begin{pmatrix} t_0 & t_{-1} & t_{-2} & \ldots & \ldots & t_{-n+1} \\ t_1 & t_0 & t_{-1} & \ddots & & \vdots \\ t_2 & t_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & t_{-1} & t_{-2} \\ \vdots & & \ddots & t_1 & t_0 & t_{-1} \\ t_{n-1} & \ldots & \ldots & t_2 & t_1 & t_0 \end{pmatrix}. \tag{4}$$

Such matrices arise in many applications. For example, suppose that $x = (x_0, x_1, \ldots, x_{n-1})^T = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$ is a column vector denoting an "input" and that t_k is zero for $k < 0$. Then the vector

$$
y = T_n x = \begin{pmatrix} t_0 & 0 & 0 & \dots & \dots & 0 \\ t_1 & t_0 & 0 & \ddots & & \vdots \\ t_2 & t_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & & \ddots & t_1 & t_0 & 0 \\ t_{n-1} & \dots & \dots & t_2 & t_1 & t_0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} x_0 t_0 \\ t_1 x_0 + t_0 x_1 \\ \sum_{i=0}^{2} t_{2-i} x_i \\ \vdots \\ \sum_{i=0}^{n-1} t_{n-1-i} x_i \end{pmatrix} \tag{5}
$$

with entries $y_k = \sum_{i=0}^{k-1} t_{k-i} x_i$ represents the the output of the discrete time causal time-invariant filter h with "impulse response" t_k. Equivalently, this is a matrix and vector formulation of a discrete-time convolution of a discrete time input with a discrete time filter.

As another example, suppose that $\{X_n\}$ is a discrete time random process with mean function given by the expectations $m_k = E(X_k)$ and covariance function given by the expectations $K_X(k, j) = E[(X_k - m_k)(X_j - m_j)]$. Signal processing theory such as prediction, estimation, detection, classification, regression, and communications and information theory are most thoroughly developed under the assumption that the mean is constant and that the covariance is Toeplitz, i.e., $K_X(k, j) = K_X(k - j)$, in which case the process is said to be weakly stationary. (The terms "covariance stationary" and "second order stationary" also are used when the covariance is assumed to be Toeplitz.) In this case the $n \times n$ covariance matrices $K_n = [K_X(k, j); k, j = 0, 1, \dots, n-1]$ are Toeplitz matrices. Much of the theory of weakly stationary processes involves applications of Toeplitz matrices. Toeplitz matrices also arise in solutions to differential and integral equations, spline functions, and problems and methods in physics, mathematics, statistics, and signal processing. A common special case of Toeplitz matrices – which will result in significant simplification and play a fundamental role in developing more general results – results when every row of the matrix is a right cyclic shift of the row above it so that $t_k = t_{-(n-k)} = t_{k-n}$ for $k = 1, 2, \dots, n-1$. In this case the picture become

$$
C_n = \begin{pmatrix} c_0 & c_{n-1} & \dots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \dots & c_1 & c_0 \end{pmatrix}. \tag{6}
$$

A matrix of this form is called a *circulant matrix* and usually is denoted by $\mathrm{circ}(c_0, c_{n-1}, \dots, c_1)$. Circulant matrices arise, for example, in applications involving the discrete Fourier transform (DFT) and the study of cyclic codes for error correction.

3. Eigenvalues, eigenvector and eigenvalue decomposition (EVD) of circulant Jacket matrices

Since in [8] it was proved that every Toplietz Jacket matrix is equivalent to circulant it is natural to look into details how the eigenvalues, eigenvectors and eigenvalue decomposition (EVD) of a circulant matrix can be calculated.

Also note that in contrast to the usual symmetric matrix S (that have the property $S = S^T$), the Toplietz matrices and circulant matrices in particular are *persymmetric* [4]. This means that a real matrix $B \in \mathbb{R}^{n \times n}$, $B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}$ is persymmetric if it's symmetric about its northeast-southwest diagonal, i.e. for all $1 \le i, j \le n$, $b_{i,j} = b_{n-j+1, n-i+1}$.

This is equivalent to requiring $B = EB^T E$, where E is the $n \times n$ exchange matrix $E = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$. **Lemma 1.** If T is a Toplietz (circulant) matrix then the following statements are satisfied:

a) T is a persymmetric matrix;

b) If T is nonsingular then its inverse T^{-1} is also persymmetric.

In contrast to diagonal matrices whose eigenvalues are their diagonal elements [9], the determination of the eigenvalues of persymmetric matrices requires tedious calculation. Their eigendecomposition depends on whether their dimension is odd or even. We will delve into the EVD of circulant matrices following [1]. Denote,

$$\Pi = \mathrm{circ}(0,1,0,\ldots,0) = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix}. \tag{7}$$

It is easy to prove that a matrix C is circulant if and only if $C.\Pi = \Pi.C$

Definition 1: Let n be a fixed integer ≥ 1 and $i = \sqrt{-1}$ is the imaginary unit. Let $\omega = e^{\frac{2\pi i}{n}} = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$. Denote

$$\Omega = (\Omega_n) = \mathrm{diag}(1, \omega, \omega^2, \ldots, \omega^{n-1}). \tag{8}$$

Theorem 1. $\Pi = F^*\Omega F$, where F is the normalized matrix of the unitary discrete Fourier Transform

$$F = \frac{1}{\sqrt{n}}(e^{-2jk\pi i/n}) = \frac{1}{\sqrt{n}} \begin{pmatrix} \omega_n^{0,0} & \omega_n^{0,1} & \ldots & \omega_n^{0,(n-1)} \\ \omega_n^{1,0} & \omega_n^{1,1} & \ldots & \omega_n^{1,(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_n^{(n-1),0} & \omega_n^{(n-1),1} & \ldots & \omega_n^{(n-1),(n-1)} \end{pmatrix}. \tag{9}$$

Using the universal circulant matrix Π and Theorem 1 we have

Theorem 2. If C is circulant it is diagonalized by F . More precisely $C = F^*\Lambda F$, where $\Lambda = \mathrm{diag}(p_\gamma(1), p_\gamma(\omega), \ldots, p_\gamma(\omega^{n-1}))$, for $p_\gamma(x) = c_0 + c_1 x + \ldots + c_{n-1}x^{n-1}$. If $\omega_j = e^{\frac{2\pi ij}{n}}$ are the n -th roots of unity the eigenvalues of C are $\lambda_j = c_0 + c_{n-1}\omega_j + c_{n-2}\omega_j^2 + \ldots + c_1\omega_j^{n-1}$, $j = 0, \ldots. n-1$ and the eigenvectors $v_j = (1, \omega_j, \omega_j^2, \ldots, \omega_j^{n-1})^T$, $j = 0,1,\ldots n-1$.

As a consequence of the explicit formula for the eigenvalues above, the determinant of circulant matrix can be computed as:

$$\det(C) = \prod_{j=0}^{n-1}(c_0 + c_{n-1}\omega_j + c_{n-2}\omega_j^2 + \ldots + c_1\omega_j^{n-1}). \tag{10}$$

4. Examples of the singular-value decomposition

In [8] there is an example of Toplietz-Jacket matrix of order 4

$$T_4(a,b,c) = \begin{pmatrix} a & b & c & -\frac{bc}{a} \\ -\frac{ab}{c} & a & b & c \\ \frac{a^2}{c} & -\frac{ab}{c} & a & b \\ \frac{a^2 b}{c^2} & \frac{a^2}{c} & -\frac{ab}{c} & a \end{pmatrix}. \tag{11}$$

Using [8, Theorem 1] we can calculate the diagonal matrices that can be used to modify the matrix to circulant Jacket.
We have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{c}}{\sqrt{a}} & 0 & 0 \\ 0 & 0 & \frac{c}{a} & 0 \\ 0 & 0 & 0 & \frac{c\sqrt{c}}{a\sqrt{a}} \end{pmatrix} T_4(a,b,c) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{a}}{\sqrt{c}} & 0 & 0 \\ 0 & 0 & \frac{a}{c} & 0 \\ 0 & 0 & 0 & \frac{a\sqrt{a}}{c\sqrt{c}} \end{pmatrix} = \begin{pmatrix} a & \frac{\sqrt{ab}}{\sqrt{c}} & a & -\frac{\sqrt{ab}}{\sqrt{c}} \\ -\frac{\sqrt{ab}}{\sqrt{c}} & a & \frac{\sqrt{ab}}{\sqrt{c}} & a \\ a & -\frac{\sqrt{ab}}{\sqrt{c}} & a & \frac{\sqrt{ab}}{\sqrt{c}} \\ \frac{\sqrt{ab}}{\sqrt{c}} & a & -\frac{\sqrt{ab}}{\sqrt{c}} & a \end{pmatrix} = C_4. \tag{12}$$

The characteristic polynomial of C is

$$\det(C - xI) = \frac{16a^3b^2}{c} - \frac{16a^2b^2}{c}x + 4a^2x^2 + \frac{4ab^2}{c}x^2 - 4ax^3 + x^4 == \left(\frac{4ab^2}{c} + x^2\right)(x^2 - 4a + 4a^2) = \left(\frac{4ab^2}{c} + x^2\right)(x - 2a)^2,$$

so the eigenvalues are $\left\{2a, 2a, -\frac{2i\sqrt{ab}}{\sqrt{c}}, \frac{2i\sqrt{ab}}{\sqrt{c}}\right\}$ and the eigenvectors are $\{(0,1,0,1),(1,0,1,0),(-i,-1,i,1),(i,-1,-i,1)\}$.

Since we do not have simple spectrum of the matrix C we proceed with singular value decomposition decomposition (SVD):

$$C_4 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2a & 0 & 0 & 0 \\ 0 & 2a & 0 & 0 \\ 0 & 0 & -\frac{2\sqrt{ab}}{\sqrt{c}} & 0 \\ 0 & 0 & 0 & \frac{2\sqrt{ab}}{\sqrt{c}} \end{pmatrix} \times \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}^T. \tag{13}$$

Another example of the Toeplitz Jacket 6×6 matrix

$$T_6 = \begin{pmatrix} 1 & \frac{1}{3} & -\frac{w}{9} & -\frac{1}{27} & \frac{1}{81} & \frac{w}{243} \\ -3w & 1 & \frac{1}{3} & -\frac{w}{9} & -\frac{1}{27} & \frac{1}{81} \\ -9 & -3w & 1 & \frac{1}{3} & -\frac{w}{9} & -\frac{1}{27} \\ 27 & -9 & -3w & 1 & \frac{1}{3} & -\frac{w}{9} \\ 81w & 27 & -9 & -3w & 1 & \frac{1}{3} \\ -243 & 81w & 27 & -9 & -3w & 1 \end{pmatrix} \tag{14}$$

can be converted to the circulant using $x = 3i$ in

$$C_6 = \operatorname{diag}(1, x^{-1}, x^{-2}, \ldots, x^{-5}) T_6 \operatorname{diag}(1, x, x^2, \ldots, x^5) == \begin{pmatrix} 1 & i & \omega & i & 1 & i\omega \\ i\omega & 1 & i & \omega & i & 1 \\ 1 & i\omega & 1 & i & \omega & i \\ i & 1 & i\omega & 1 & i & \omega \\ \omega & i & 1 & i\omega & 1 & i \\ i & \omega & i & 1 & i\omega & 1 \end{pmatrix} = \operatorname{circ}(1, i, \omega, i, 1, i\omega),$$

where $\omega = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

The SVD in this case is

$$C_6 = \begin{pmatrix} i\omega & 1 & i & \omega & i & 1 \\ 1 & i & \omega & i & 1 & i\omega \\ i & \omega & i & 1 & i\omega & 1 \\ \omega & i & 1 & i\omega & 1 & i \\ i & 1 & i\omega & 1 & i & \omega \\ 1 & i\omega & 1 & i & \omega & i \end{pmatrix} I_6 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{15}$$

5. Conclusions

In this paper we investigate the circulant Jacket matrices that have a similar structure of a special kind of Toeplitz and conventional circulant matrices. The extensive benefits of these CJ matrices are the very fast and spontaneously computation of inverse matrices. The fact that CJ matrices are diagnosable by fast Fourier transform matrices and that they possess nice singular-value decomposition make them very important. The CJ matrices can also be used in random channels for mobile communication and signal processing.

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