Bounds on the Maximum Determinant for $(1, -1)$ Matrices

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Abstract

We suppose the Hadamard conjecture is true and an Hadamard matrix of order $4t$, exists for all $t \geq 1$. We use the results for the equivalent $SBIBD(4t-1, 2t-1, t-1)$ to establish the maximum determinant or a lower bound for the maximum determinant for all ± 1 matrices. In particular we give numerical results for all orders ≤ 100.

Key Words and Phrases: Maximum determinant, SBIBD, incidence matrix, minors, bounds.
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1 Introduction

Let $\mathcal{X}_n$ be the set of all ± 1 matrices of order n. The question of the maximal value of the determinant of an element $X \in \mathcal{X}_n$ is an old one which goes back to the beginnings of matrix theory. It is a simple consequence of Hadamard's inequality [13] that for all $X \in \mathcal{X}_n$

$$det(X) \leq (\prod_{i=1}^{n} \sum_{j=1}^{n} x_{ij}^2)^{\frac{1}{2}} \leq n^{\frac{n}{2}}. \quad (1)$$

There is a large body of work addressing the question of when (1) is sharp. Matrices in $\mathcal{X}_n$ for which equality holds in (1) are known as Hadamard matrices, H, which satisfy $HH^T = H^T H = nI_n$. For an $n \times n$ Hadamard matrix to exist it is necessary that n be either $1, 2$ or $n \equiv 0 (mod\ 4)$, and it is conjectured that this condition is also sufficient. According to [20] the smallest value for which the existence of an Hadamard matrix is in question is $n = 4 \cdot 107 = 428$. For $n \equiv 1 (mod\ 4)$ it was proved by Ehlich [8] that for all $X \in \mathcal{X}_n$,

$$det(X) \leq (2n-1)^{\frac{1}{2}} (n-1)^{\frac{(n-1)}{2}} \quad (2)$$

and in order for equality to hold in (7) it is necessary that $2n - 1$ be a square and that there exists an $X \in \mathcal{X}_n$ with $XX^T = (n-1)I_n + J_n$, where J_n is the $n \times n$ matrix whose all entries are equal to one, and I_n is the $n \times n$ identity matrix. For $n \equiv 2 (mod\ 4)$ Ehlich [8], and independently Wojtas [24], proved that for all $X \in \mathcal{X}_n$,

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$$det(X) \leq (2n-2)(n-2)^{\frac{n}{2}-1}. \tag{3}$$

Moreover, the equality in (3) holds if and only if there exists $X \in \mathcal{X}_n$ such that

$$XX^T = X^T X = \begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix},$$

where $L = (n-2)I + 2J$ is an $\frac{n}{2} \times \frac{n}{2}$ matrix. A further necessary condition for equality to hold is that $2n-2$ is the sum of two squares. J.H.E. Cohn [6] gave an independent proof of this result and provided further information on the structure of maximal examples. Ehlich [9] investigated the case $n \equiv 3(mod\,4)$ which appears the most difficult case. Assume $n \equiv 3(mod\,4)$ and $n \geq 63$. Ehlich [9] proved that for all $X \in \mathcal{X}_n$,

$$det(X) \leq (\frac{4 \cdot 11^6}{7^7}(n-3)^{n-7}n^7)^{\frac{1}{2}}. \tag{4}$$

Moreover, for the equality to hold it is necessary that $n = 7m$ and that there exists $X \in \mathcal{X}_n$ with

$$XX^T = I_7 \otimes [(n-3)I_m + 4J_m] - J_n.$$

The corresponding bounds for all values $n \equiv 3(mod\,4)$, $n < 63$, are also given in [9], as are structures of XX^T for normalized maximal examples X. The formula for values $n < 63$ is the same as for the above example. A ± 1 matrix X has maximal determinant if XX^T has block structure with the blocks along the diagonal of the form $(n-3)I + 3J$ and the off-diagonal blocks equal to $-J$.

Though it is well known that the Hadamard bound in the case $n \equiv 0(mod\,4)$ is attained infinitely often and has to be considered sharp in this sense, it was not known if the bounds given in (2), (3) and (4) are sharp in this sense.

In this paper we report that the results of [16] may be applied to the $SBIBD(4t-1, 2t-1, t-1)$ to give us a lower bound for infinitely many ± 1 matrices. However, we believe that for a particular value of $n \equiv 1, 2$ or $3(mod\,4)$ for which the upper bound given in (2), (3) and (4) cannot be attained, an efficient computer search is likely to produce examples whose determinants have larger values that the ones given here by us. The real challenge, however, is to find an infinite family of examples whose determinants take on the bound in (4) or to show that the bound in (4) can be improved. It is conceivable that it is not sharp.

If X is a design of order n with elements ± 1 and X^* is the D-optimal design of the same order we define the efficiency of X by the ratio $det(X)/det(X^*)$.

It is obvious that for $n =$ 22,34,58,70,78,94 ($n \leq 100$) the upper bound given in (3) cannot be attained as $n-1$ is not the sum of two squares. Thus, we discuss the efficiency of some designs which we give here. For example, for $n = 34$ the upper bound given in (3) is $32^{16} \cdot 66$. From our theorems we can obtain a design of order 34 with determinant $36^{16} \cdot 2$. Hence, the efficiency of this design is larger than 0.20 which, however, is a very small efficiency. For $n = 58$ the upper bound given in (3) is $56^{28} \cdot 114$. ¿From our theorems we can obtain a design of order 58 with determinant $60^{28} \cdot 2$. Hence, the efficiency of this design is larger than 0.12 which, however, is is a very small efficiency. For $n = 70$ the upper bound given in (3) is $68^{34} \cdot 138$. ¿From our theorems we can obtain a design of order 70 with determinant $72^{34} \cdot 2$. Hence, the efficiency of this design is larger than 0.10 which, however, is also a very small efficiency.

Cohn [7] found almost D-optimal designs of orders $15, 19, 22$, which have determinants $2^{14} \cdot 25515 = 2^{14} \cdot 3^6 \cdot 5 \cdot 7$, $2^{18} \cdot 3411968 = 2^{30} \cdot 7^2 \cdot 17$, and $2^{21} \cdot 184769649 = 2^{21} \cdot 3^2 \cdot 23^2 \cdot 197^2$ respectively. Although these designs have not been proved to be optimal we note they have very high efficiency being larger than $0.97, 0.975$ and 0.90 respectively.

In the present paper we use the maximum determinant for the $(1, -1)$ incidence matrices of certain SBIBDs to obtain lower bounds for all orders n.

For the purpose of this paper we will define a SBIBD(v, k, λ) to be a $v \times v$ matrix, B, with entries 0 or 1, which has exactly k entries $+1$ and $v - k$ entries 0 in each row and column and for which the inner product of any distinct pairs of rows and columns is λ. The $(1, -1)$ incidence matrix of B is obtained by letting $A = 2B - J$, where J is the $v \times v$ matrix with entries all $+1$. We write I for the identity matrix of order v.

Then we have

$$BB^T = (k - \lambda)I + \lambda J \tag{5}$$

and its equivalent ± 1 matrix satisfies

$$AA^T = 4(k - \lambda)I + (v - 4(k - \lambda))J. \tag{6}$$

The determinant simplification theorem in [16] shows that

$$detB = (k - \lambda)^{\frac{v-1}{2}} \sqrt{k + (v - 1)\lambda}$$

and since $\lambda(v - 1) = k^2 - k$

$$detA = 2^{v-1}(k - \lambda)^{\frac{v-1}{2}} |v - 2k| \tag{7}$$

or with $x = v - 4k + 4\lambda$,

$$detA = (v - x)^{\frac{1}{2}(v-1)} |v - 2k|.$$

A D–optimal design of order n is an $n \times n$ matrix with entries ± 1 having maximum determinant. For orders $n \equiv 0 (\text{mod } 4)$ the matrix with entries ± 1 and maximal determinant is an *Hadamard matrix, H.*

2 Minors of the ± 1 Incidence Matrix of an SBIBD

We note from Koukouvinos, Mitrouli and Seberry [16] the maximal value of the $v \times v$, $(v - 1) \times (v - 1)$ and $(v - 2) \times (v - 2)$ minors for some $SBIBD(v, k, \lambda)$ are as given in Table 1, and Theorem 1 below.

Theorem 1 *Write $x = v - 4(k - \lambda)$.*

1) The $(v - 1) \times (v - 1)$ minors of the $(1, -1)$ incidence matrix of an $SBIBD(v, k, \lambda)$, A, have value

	Minor of $(2s^2+2s+1, s^2, \lambda)$ $\lambda = \frac{1}{2}(s^2-s))$	Minor of $(4s^2, 2s^2+s, s^2+s)$	Minor of $(4t-1, 2t-1, t-1)$
$v \times v$	$(2s+1)(2s^2+2s)^{s^2+s}$	$(4s^2)^{2s^2}$	$(4t)^{2t-1}$
$(v-1)\times(v-1)$	$2(s+1)(2s^2+2s)^{s^2+s-1}$	$(4s^2)^{2s^2-1}$	$2(4t)^{2t-2}$
$(v-2)\times(v-2)$	$4(s+1)(2s^2+2s)^{s^2+s-2}$	$2(4s^2)^{2s^2-2}$	$4(4t)^{2t-3}$

Table 1: Values of Large Minors of Some SBIBD

$$(v-x)^{\frac{1}{2}(v-3)}\sqrt{(v-2k\pm 1)^2 x + (x-1)(-vx-v+2x)}. \tag{8}$$

This gives maximum determinant

$$(v-x)^{\frac{1}{2}(v-3)}\sqrt{x(v-2k+1)^2 + (x-1)(-vx-v+2x)}.$$

2) The $(v-2)\times(v-2)$ *minors of the* $(1,-1)$ *incidence matrix of an SBIBD*$(v,\ k,\ \lambda)$*,* A*, have value equal to*

$$2(v-x)^{\frac{1}{2}(v-5)}\sqrt{x(v-2k\pm 1)^2 + (x-1)(-vx-v+2x)}.$$

This gives maximum determinant

$$2(v-x)^{\frac{1}{2}(v-5)}\sqrt{x(v-2k+1)^2 + (x-1)(-vx-v+2x)}.$$

□

Theorem 2 *Suppose* $4t$ *is the order of an Hadamard matrix. Write* $v = 4t-1$*. Then there are* ± 1 *matrices whose*

- $v \times v$ *determinants have magnitude* $(4t)^{2t-1}$*;*
- $(v-1)\times(v-1)$ *determinants have magnitude* $2(4t)^{2t-2}$*;*
- $(v-2)\times(v-2)$ *determinants have magnitude* $4(4t)^{2t-3}$*.*

Proof. We note that every Hadamard matrix of order $4t$ is equivalent to an $SBIBD(4t-1, 2t-1, t-1)$. The results above for the $SBIBD(4t-1, 2t-1, t-1)$ give us lower bounds. □

These results, writing $v = 4t-1$, give us the lower bounds for

- $v \times v$ determinants of $(4t)^{2t-1}$ for $t \geq 0$;
- $(v-1)\times(v-1)$ determinants of $2(4t)^{2t-2}$ for $4t-1 = 23$ 35, 59, 71 and 79;

- $(v-2)\times(v-2)$ determinants of $4(4t)^{2t-3}$ for $4t-1 = 31, 35, 39, 47, 51, 55, 59, 67, 71, 75, 79, 83, 87, 91, 95$ and 99.

These results are improved from $\geq 16^7$ for $4t-1=15$ to $d_{15} \geq 2^{14}3^6 35$ in [7]; from $\geq 20^9$ for $4t-1=19$ to $d_{19} \geq 2^{30}7^2 17$ in [7]; from $2 \cdot 24^{10}$ for $4t-2=22$ to $d_{22} \geq 2^{21}3^2 23^2 197^2$ in [7]; and from $\geq 2^{67}$ for $4t-3=29$ to $d_{29} \geq 2^{28}7^{13}43 = (28)^{13}172$ in [15]. The result for $4t-3=17$ is given in [17] and for $4t-3=21$ is given in [5].

In Table 2 we give bounds for the maximum determinant d_n of all $n \times n$ matrices with elements ± 1, for $n \leq 100$.

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n	d_n	Ref	n	d_n	Ref
1	1	[13]	51	$\geq 52^{25}$	$\surd$
2	2	[13]	52	52^{26}	[18]
3	4	[13]	53	$\geq 52^{26} \cdot 8$	[11]
4	4^2	[13]or[21]	54	$52^{26} \cdot 106$	[26]
5	$4^2 \cdot 3$	[19]	55	$\geq 56^{27}$	$\surd$
6	$4^2 \cdot 10$	[8]	56	56^{28}	[18]
7	$2^6 \cdot 9$	[23]	57	$\geq 56^{27} \cdot 456$	[11]
8	8^4	[21]	58	$\geq 60^{28} \cdot 2$	$\surd$
9	$2^9 \cdot 28$	[10]	59	$\geq 60^{29}$	$\surd$
10	$8^4 \cdot 18$	[8]	60	60^{30}	[18]
11	$2^{16} \cdot 5$	[12]	61	$60^{30} \cdot 11$	[3]
12	12^6	[13]	62	$60^{30} \cdot 122$	[28]
13	$12^6 \cdot 5$	[19]	63	$\geq 64^{31} \cdot 8$	$\surd$
14	$12^6 \cdot 26$	[8]	64	64^{32}	[21]
15	$\geq 2^{14} \cdot 3^6 \cdot 35$	[7]	65	$\geq 64^{32} \cdot 9$	[11]
16	16^8	[21]	66	$64^{32} \cdot 130$	[29]
17	$16^7 \cdot 80$	[17]	67	$\geq 68^{33}$	$\surd$
18	$16^8 \cdot 34$	[8]	68	68^{34}	[18]
19	$\geq 2^{30} \cdot 7^2 \cdot 17$	[7]	69	$\geq 68^{34} \cdot 9$	[11]
20	20^{10}	[13]	70	$\geq 72^{34} \cdot 2$	$\surd$
21	$20^9 \cdot 116$	[5]	71	$\geq 72^{35}$	$\surd$
22	$\geq 2^{21} \cdot 3^2 \cdot 23^2 \cdot 197^2$	[7]	72	72^{36}	[18]
23	$\geq 24^{11}$	$\surd$	73	$\geq 72^{36} \cdot 9$	[11]
24	24^{12}	[18]	74	$72^{36} \cdot 146$	[6]
25	$24^{12} \cdot 7$	[19]	75	$\geq 76^{37}$	$\surd$
26	$24^{12} \cdot 50$	[8]	76	76^{38}	[18]
27	$\geq 28^{13}$	$\surd$	77	$\geq 76^{37} \cdot 696$	[11]
28	28^{14}	[18]	78	$\geq 80^{38} \cdot 2$	$\surd$
29	$\geq 28^{13} \cdot 172$	[15]	79	$\geq 80^{39}$	$\surd$
30	$28^{14} \cdot 58$	[8]	80	80^{40}	[18]
31	$\geq 2^{75}$	$\surd$	81	$\geq 80^{39} \cdot 784$	[11]
32	2^{80}	[21]	82	$80^{40} \cdot 162$	[14]or[6]
33	$\geq 36^{15} \cdot 4$	$\surd$	83	$\geq 84^{41}$	$\surd$
34	$\geq 36^{16} \cdot 2$	$\surd$	84	84^{42}	[18]
35	$\geq 36^{17}$	$\surd$	85	$\geq 84^{42} \cdot 10$	[11]
36	36^{18}	[18]	86	$84^{42} \cdot 170$	[4]
37	$\geq 36^{18} \cdot 7$	[11]	87	$\geq 88^{43}$	$\surd$
38	$36^{18} \cdot 74$	[8]	88	88^{44}	[18]
39	$\geq 40^{19}$	$\surd$	89	$\geq 88^{44} \cdot 10$	[11]
40	40^{20}	[18]	90	$88^{44} \cdot 178$	[6]
41	$40^{20} \cdot 9$	[2]or[22]	91	$\geq 92^{45}$	$\surd$
42	$40^{20} \cdot 82$	[25]	92	92^{46}	[1]
43	$\geq 44^{21}$	$\surd$	93	$\geq 92^{46} \cdot 10$	[11]
44	44^{22}	[18]	94	$96^{46} \cdot 2$	$\surd$
45	$\geq 44^{21} \cdot 324$	[11]	95	$\geq 96^{47}$	$\surd$
46	$44^{22} \cdot 90$	[25]	96	96^{48}	[18]
47	$\geq 48^{23}$	$\surd$	97	$\geq 96^{47} \cdot 1016$	[11]
48	48^{24}	[18]	98	$96^{48} \cdot 194$	[6]
49	$\geq 48^{23} \cdot 372$	[11]	99	$\geq 10 \cdot 100^{49}$	$\surd$
50	$48^{24} \cdot 98$	[28]	100	100^{50}	[18]

Table 2: $\surd$ means given here for the first time

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