

A SKEW HADAMARD MATRIX OF ORDER 36

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Hadamard matrices exist for infinitely many orders $4m$, $m \geq 1$, m integer, including all $4m < 100$, cf. [3], [2]. They are conjectured to exist for all such orders. Skew Hadamard matrices have been constructed for all orders $4m < 100$ except for 36, 52, 76, 92, cf. the table in [4]. Recently Szekeres [6] found skew Hadamard matrices of the order $2(p^t + 1) \equiv 12 \pmod{16}$, p prime, thus covering the case 76. In addition, Blatt and Szekeres [1] constructed one of order 52. The present note contains a skew Hadamard matrix of order 36 (and one of order 52), thus leaving 92 as the smallest open case.

The unit matrix of any order is denoted by I . The square matrices Q and R of order m are defined by their only nonzero elements

$$q_{i,i+1} = q_{m,1} = 1, \quad i = 1, \dots, m-1; \quad r_{i,m-i+1} = 1, \quad i = 1, \dots, m$$

We have

$$Q^m = I, \quad R^2 = I, \quad RQ = Q^T R.$$

Any square matrix A of order m is symmetric if $A = A^T$, skew if $A + A^T = 0$, circulant if $AQ = QA$. Hence, for circulant A we have

$$A = \sum_{i=0}^{m-1} a_i Q^i, \quad RA = A^T R.$$

Any square matrix H of order $4m$ is *skew Hadamard* if its elements are 1 and -1 (we write $+$ and $-$) and

$$HH^T = 4mI, \quad H + H^T = 2I.$$

THEOREM 1. *If A, B, C, D are square circulant matrices of order m , if A is skew, and if*

$$AA^T + BB^T + CC^T + DD^T = (4m-1)I,$$

then

$$H = \begin{bmatrix} A+I & BR & CR & DR \\ -BR & A+I & -D^T R & C^T R \\ -CR & D^T R & A+I & -B^T R \\ -DR & -C^T R & B^T R & A+I \end{bmatrix}$$

satisfies $HH^T = 4mI, H + H^T = 2I$.

PROOF. By straightforward verification.

REMARK. If, in addition, $B, C,$ and D are symmetric, then H may be written in terms of the quaternion matrices K_4, L_4, M_4 and the Kronecker product \otimes as follows:

$$H = I_4 \otimes (I + A) + K_4 \otimes BR + L_4 \otimes CR + M_4 \otimes DR,$$

hence looking much like a Williamson-type matrix, cf. [7].

THEOREM 2. *There exist skew Hadamard matrices of orders 36 and 52.*

PROOF. We apply theorem 1 with the following circulant matrices of order 9:

$$\begin{aligned} A &= (0 \ + \ + \ - \ + \ - \ + \ - \ -), \quad B = (+ \ - \ + \ + \ - \ - \ + \ + \ -), \\ C &= (- \ - \ + \ + \ + \ + \ + \ + \ -), \quad D = (+ \ + \ + \ - \ + \ + \ - \ + \ +). \end{aligned}$$

By inspection the skew A and the symmetric B, C, D are seen to satisfy the hypotheses. Hence a skew Hadamard matrix of order 36 is obtained. Secondly, we consider the following circulant matrices of order 13:

$$\begin{aligned} A &= (0 \ + \ + \ + \ - \ + \ + \ - \ - \ + \ + \ - \ -), \\ B &= (- \ + \ - \ + \ + \ - \ - \ - \ - \ + \ + \ - \ +), \\ C = D &= (- \ - \ + \ - \ + \ + \ + \ + \ + \ - \ + \ + \ +). \end{aligned}$$

Application of theorem 1 to A, B, C, D yields a skew Hadamard matrix of order 52 since

$$AA^T = 15I - J + 2B, \quad BB^T = 12I - J - 2B, \quad CC^T = DD^T = 12I + J.$$

REMARK. The positive elements of B indicate the quadratic residues mod 13. The matrix of order 26

$$\begin{bmatrix} B+I & C \\ C^T & -B-I \end{bmatrix}$$

is an orthogonal matrix with zero diagonal, cf. [2] p. 1007. The matrix A describes the unique tournament of order 13 having no transitive subtournament of order 5, which was recently found by Reid and Parker [5].

References

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