Toeplitz jacket matrices are circulant

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In this paper we prove that all Toeplitz jacket matrices are circulant, up to equivalence. As a corollary we show that a Toeplitz real Hadamard matrix is either circulant or negacyclic.

Introduction: A Toeplitz matrix is an $n \times n$ matrix T, where $t_{i,j} = t_{j-i}$, for every $1 \leq i, j \leq n$, i.e. a matrix of the form

$$T = \begin{bmatrix} t_0 & t_1 & t_2 & \dots & t_{n-1} \\ t_{-1} & t_0 & t_1 & & \\ t_{-2} & t_{-1} & t_0 & & \vdots \\ \vdots & & & \ddots & \\ t_{-n+1} & & \dots & & t_0 \end{bmatrix}. \quad (1)$$

Such matrices arise in many applications [1]. We mention here two special kind of Toeplitz matrices as follows. One important class is formed by circulant matrices, where each row vector is rotated one element to the right relative to the preceding row vector. In particular, with the notations of (1) we have $t_i = t_{-n+i}$, for each $i = 1, 2, \dots n-1$ within circulant matrices. Another class of Toeplitz matrices are negacyclic matrices, in which $t_i = -t_{-n+i}$ holds for each $i = 1, 2, \dots, n-1$. For the general treatment of Toeplitz matrices we refer the reader to [2].

In this note we investigate the existence of Toeplitz jacket matrices. A matrix M of order n is called jacket [3] (also called within the mathematics community as type II, [4]; or inverse orthogonal [5]), if its inverse satisfies $[M^{-1}]_{i,j} = \frac{1}{n m_{j,i}}$, i.e. the inverse matrix can be obtained by taking element-wise inverse and transposition, up to a negligible constant factor. Equivalently, jacket matrices satisfy the following relations:

$$\sum_{k=1}^{n} \frac{m_{i,k}}{m_{j,k}} = n\delta_{ij}, \qquad i, j = 1, 2, \dots, n. \quad (2)$$

Note that the jacket matrices considered in this letter do not have any zero entries. A jacket matrix in which all entries are of modulus 1 is called a complex Hadamard matrix [6, 7]. It is easy to see that if K is a jacket matrix, then for every permutation matrices P_1, P_2 and for every invertible diagonal matrices D_1, D_2 the matrix $H = P_1 D_1 K D_2 P_2$ is a jacket matrix as well. Jacket matrices related in this fashion are called equivalent. Finding all jacket matrices up to equivalence turns out to be a challenging problem, and has been solved only up to orders $n \leq 5$ [4].

Circulant jacket matrices have been heavily investigated during the early 1990s by Björck and coauthors, who were motivated by finding all solutions to the so-called cyclic n-roots problem (see [10] and the references therein).

Jacket matrices have both theoretical applications ranging from harmonic analysis [8] to quantum information theory [5]; as well as applications in signal processing [3, 9].

We are interested in the Toeplitz jacket case because our aim is to obtain new examples of complex Hadamard matrices. However, it turns out that the jacket property puts heavy restrictions onto the structure of Toeplitz matrices, and we prove, in Section 2, the following rather surprising result.

Theorem 1: Every Toeplitz jacket matrix is equivalent to a circulant one.

Circulant and Toeplitz jacket matrices: First we provide the reader with some examples. The following is well-known, see e.g. [10].

Example 1: Let $n \geq 2$ be an integer, and let $\mathbb{Z}_n^*$ denote the set of invertible elements in the ring $\mathbb{Z}_n$. For $(\alpha, \beta) \in \mathbb{Z}_n^* \times \mathbb{Z}_n$ consider the row vector $x(\alpha, \beta)$ of length n with entries

$$[x(\alpha,\beta)]_i = \begin{cases} \exp\left(\frac{2\pi \mathbf{i}}{n}\left(\frac{\alpha i^2}{2} + \beta i\right)\right), & i \in \mathbb{Z}_n, \quad n \text{ even}, \\ \exp\left(\frac{2\pi \mathbf{i}}{n}\left(\frac{\alpha i(i-1)}{2} + \beta i\right)\right), & i \in \mathbb{Z}_n, \quad n \text{ odd}, \end{cases}$$

where $\mathbf{i}$ is the complex imaginary unit. Then the circulant complex Hadamard matrix whose first row is $x(\alpha, \beta)$ is equivalent to the discrete Fourier transform (DFT) matrix.

The "Potts model" [11] is another class of circulant jacket matrices.

Example 2: Let us denote by I_n and J_n the identity matrix and the matrix of all 1s of order n, respectively. Further, let α_n be any root of the quadratic equation $\alpha^2 + (n-2)\alpha + 1 = 0$. Then, the matrix $P = (\alpha_n - 1)I_n + J_n$ is a circulant jacket matrix. Note that for $n > 4$ the matrix arising from P does not have unimodular entries and hence it is not a complex Hadamard matrix.

Finally, we provide examples of order $n = 4$ as follows.

Example 3: Let a, b, c be nonzero complex numbers. Then the matrices, arising of the form below are all Toeplitz jacket matrices of order 4:

$$T_4(a,b,c) = \begin{bmatrix} a & b & c & -\frac{bc}{a} \\ -\frac{ab}{c} & a & b & c \\ \frac{a^2}{c} & -\frac{ab}{c} & a & b \\ \frac{a^2 b}{c^2} & \frac{a^2}{c} & -\frac{ab}{c} & a \end{bmatrix}.$$

Next we give a method to construct new Toeplitz matrices from old ones. We denote by $\mathrm{Diag}(a_1, a_2, \dots, a_n)$ the $n \times n$ diagonal matrix with diagonal entries a_i, $i = 1, 2, \dots, n$.

Lemma 1: Let T be a Toeplitz matrix of order n, and let a be an arbitrary, while b be a nonzero complex number. Then the following is a Toeplitz matrix as well:

$$T' = a\mathrm{Diag}(1, b^{-1}, b^{-2}, \dots, b^{-n+1}) T \mathrm{Diag}(1, b, b^2, \dots, b^{n-1}).$$

Proof: It is easy to see that the (i,j)th entry of T' reads $t_{i,j} a b^{j-i}$. ■

Now we state a structural result concerning Toeplitz jacket matrices.

Proposition 1: Suppose that T is a Toeplitz jacket matrix. Then, with the notations from (1)*, we have*

$$t_{-\ell} = \frac{t_{-1}}{t_{n-1}} t_{n-\ell}, \qquad \ell = 1, 2, \dots, n-1. \quad (3)$$

Proof: Note that formula (3) holds trivially for $\ell = 1$. We begin the proof by showing that it holds for $\ell = 2$ as well. We assume that $n \geq 3$.

Consider condition (2) within the first two rows of T, i.e. for $(i, j) = (2, 1)$. We have

$$\frac{t_{-1}}{t_0} + \frac{t_0}{t_1} + \dots + \frac{t_{n-2}}{t_{n-1}} = 0. \quad (4)$$

Next consider (2) for the pair of rows $(i, j) = (3, 2)$, and derive

$$\frac{t_{-2}}{t_{-1}} + \frac{t_{-1}}{t_0} + \frac{t_0}{t_1} + \dots + \frac{t_{n-3}}{t_{n-2}} = 0.$$

Add $\frac{t_{n-2}}{t_{n-1}}$ to both sides and use (4) to conclude that (3) holds for $\ell = 2$ as well. In particular, we arrived at case $\ell = 1$ of the following:

$$\frac{t_{-\ell-1}}{t_{-\ell}} = \frac{t_{n-\ell-1}}{t_{n-\ell}}, \qquad \ell = 1, 2, \dots, n-2. \quad (5)$$

Now we use mathematical induction to prove that (5) holds for all $1 \leq \ell \leq n-2$. We can assume that $n \geq 4$ and (5) already holds for some $\ell \leq n-3$. Then, we consider condition (2) for the pair of rows $(\ell+3, \ell+2)$:

$$\sum_{k=1}^{n} \frac{t_{-\ell-2+k-1}}{t_{-\ell-1+k-1}} = 0,$$

and rewrite it, using (5), to

$$\frac{t_{-\ell-2}}{t_{-\ell-1}} + \sum_{\substack{k=1 \\ k \neq n-\ell}}^{n} \frac{t_{k-2}}{t_{k-1}} = 0.$$

Adding the missing term $t_{n-\ell-2}/t_{n-\ell-1}$ to both sides, and reducing via (4) concludes the validity of (5) for $\ell + 1$.

We finish the proof by observing that (5) connects the consecutive terms $t_{-\ell}$ and $t_{-\ell-1}$ and hence

$$t_{-\ell-1} = t_{-\ell} \frac{t_{n-\ell-1}}{t_{n-\ell}} = \frac{t_{-1}}{t_{n-1}} t_{n-\ell} \frac{t_{n-\ell-1}}{t_{n-\ell}} = \frac{t_{-1}}{t_{n-1}} t_{n-\ell-1}.$$

■

Now we are ready to prove Theorem 1 rigorously.

Proof of Theorem 1: Let T be a Toeplitz jacket matrix as in (1), and let $x = \sqrt[n]{t_{-1}/t_{n-1}}$, where the operator $\sqrt[n]{.}$ denotes the principal nth root. By Lemma 1 the matrix

$$C = \mathrm{Diag}\left(1, x^{-1}, x^{-2}, \ldots, x^{-n+1}\right) T\mathrm{Diag}\left(1, x, x^2, \ldots, x^{n-1}\right)$$

is a Toeplitz matrix, which is equivalent to T. We claim that C is a circulant jacket matrix. To see this, it is enough to show that $c_{1,j} = c_{n-j+2,1}$ for every $j = 2, \ldots, n$. It is clear that $c_{1,j} = t_{j-1}x^{j-1}$. On the other hand, by using Proposition 1, we have

$$c_{n-j+2,1} = t_{-n+j-1}x^{-n+j-1} = \frac{t_{-1}}{t_{n-1}} t_{j-1}x^{-n+j-1} = t_{j-1}x^{j-1}.$$

■

Next we briefly discuss Toeplitz real Hadamard matrices.

Corollary 1: Let H be a Toeplitz real Hadamard matrix. Then H is either circulant, or negacyclic.

Proof: The result comes from Proposition 1 as follows: if the quotient $t_{-1}/t_{n-1} = 1$ then, by formula (3) we have $t_{-\ell} = t_{n-\ell}$ for $\ell = 1, 2, \ldots n-1$, i.e. the matrix is circulant. Otherwise $t_{-1}/t_{n-1} = -1$ and hence $t_{-\ell} = -t_{n-\ell}$, i.e. the matrix is negacyclic. ■

It is conjectured that for $n > 4$ there are no real circulant Hadamard matrices [6].

Example 4: The following are the only known Toeplitz real Hadamard matrices, up to equivalence:

$$H_1 = \begin{bmatrix} 1 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad H_4 = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}.$$

Conclusion: In this letter we have proved that the concept of Toeplitz and circulant jacket matrices are the same, up to equivalence.

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