

ON BASE AND TURYN SEQUENCES

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ABSTRACT. Base sequences of lengths $n + 1$, $n + 1$, n , n are constructed for all decompositions of $4n + 2$ into four squares for $n = 19, \dots, 24$. The construction is achieved through an algorithm which is also presented. It is proved through an exhaustive search that Turyn sequences do not exist for $n = 18, \dots, 27$; since Turyn sequences cannot exist for $n = 28$ or 29 , the first unsettled case is $n = 30$.

1. INTRODUCTION

Given the sequence $A = \{a_1, \dots, a_n\}$ of length n , the nonperiodic autocorrelation function $N_A(s)$ is defined as

$$(1) \quad N_A(s) = \sum_{i=1}^{n-s} a_i a_{i+s}, \quad s = 0, 1, \dots, n-1.$$

If $A(z) = a_1 + a_2 z + \dots + a_n z^{n-1}$ is the polynomial associated with the sequence A , then

$$(2) \quad A(z)A(z^{-1}) = \sum_{i=1}^n \sum_{j=1}^n a_i a_j z^{i-j} = N_A(0) + \sum_{s=1}^{n-1} N_A(s)(z^s + z^{-s}), \quad z \neq 0.$$

If $A^* = \{a_n, \dots, a_1\}$ is the reversed sequence, then

$$(3) \quad A^*(z) = z^{n-1} A(z^{-1}).$$

Base and Turyn sequences are finite sequences, with zero autocorrelation function, useful in constructing orthogonal designs and Hadamard matrices [3], in communications engineering [5], etc.

Definition 1. The four sequences X, Y, Z, W of length n , with entries $0, 1, -1$, are called T -sequences if

$$(4) \quad \begin{aligned} & \text{(i) } |x_i| + |y_i| + |z_i| + |w_i| = 1, \quad i = 1, \dots, n, \\ & \text{(ii) } N_X(s) + N_Y(s) + N_Z(s) + N_W(s) = \begin{cases} 0, & s = 1, \dots, n-1, \\ n, & s = 0. \end{cases} \end{aligned}$$

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Yang [6] gives another name for T -sequences and calls them four-symbol δ -codes; he also calls the quadruple $Q = X + Y$, $R = X - Y$, $S = Z + W$, $T = Z - W$ a regular δ -code of length n , where X , Y , Z , W are T -sequences of length n .

If Williamson type matrices of size w exist, and T -sequences of length n exist, then Hadamard matrices of size $4nw$ can be constructed (Cooper and (Seberry) Wallis [2]).

If $X(z)$, $Y(z)$, $Z(z)$, $W(z)$ are the associated polynomials, then from Definition 1 and (2) we see that (ii) in (4) can be replaced by

$$(5) \quad X(z)X(z^{-1}) + Y(z)Y(z^{-1}) + Z(z)Z(z^{-1}) + W(z)W(z^{-1}) = n, \quad z \neq 0.$$

In §2 we develop an algorithm and we construct base sequences of lengths $n+1$, $n+1$, n , n for $n = 19, \dots, 24$ for all decompositions of $4n+2$ into four squares. These are given in Table 1. In §3 we use an exhaustive search to prove that Turyn sequences of lengths $n+1$, $n+1$, n , n , do not exist for $n = 18, \dots, 27$.

2. BASE SEQUENCES

Definition 2. The four sequences A , B , C , D of lengths $n+p$, $n+p$, n , n , with entries $+1$, -1 , are called base sequences if

$$(6) \quad \begin{cases} N_A(s) + N_B(s) + N_C(s) + N_D(s) = \begin{cases} 0, & s = 1, \dots, n-1, \\ 4n+2p, & s = 0, \end{cases} \\ N_A(s) + N_B(s) = 0, & s = n, \dots, n+p-1. \end{cases}$$

Equivalently, (6) can be replaced by (see (2))

$$(7) \quad A(z)A(z^{-1}) + B(z)B(z^{-1}) + C(z)C(z^{-1}) + D(z)D(z^{-1}) = 4n+2p, \quad z \neq 0,$$

where $A(z)$, $B(z)$, $C(z)$, $D(z)$ are the associated polynomials. Base sequences of lengths $n+p$, $n+p$, n , n , whenever $p = 1$, are denoted by $BS(2n+1)$.

From (7), and for $p = 1$, if we set $z = 1$, we obtain

$$(8) \quad a^2 + b^2 + c^2 + d^2 = 4n+2,$$

where a , b , c , d are the sum of the elements of A , B , C , D , respectively. $BS(2n+1)$ for all decompositions of $4n+2$ into four squares for $n = 1, \dots, 18$ are given in [1, 4]. Also, $BS(2n+1)$ for $n = 19, 20, \dots, 23, 25, 26, 29$ and $n = 2^a 10^b 26^c$ (Golay numbers), a , b , c nonnegative integers, are given in Yang [7].

In Table 1 we give $BS(2n+1)$ for all decompositions of $4n+2$ into four squares for $n = 19, \dots, 24$.

Theorem 1. If $A = \{a_1, \dots, a_{n+1}\}$, $B = \{b_1, \dots, b_{n+1}\}$, $C = \{c_1, \dots, c_n\}$, $D = \{d_1, \dots, d_n\}$ are $\text{BS}(2n+1)$, then

$$(9) \quad \begin{aligned} a_s + b_s + a_{n+2-s} + b_{n+2-s} &\equiv \begin{cases} 2 \pmod{4}, & s = 1, \\ 0 \pmod{4}, & s = 2, \dots, [(n+1)/2], \end{cases} \\ c_s + d_s + c_{n+1-s} + d_{n+1-s} &\equiv 0 \pmod{4}, \quad s = 1, \dots, [n/2]. \end{aligned}$$

Proof. We have

$$\begin{aligned} N_A(s) + N_B(s) &= \sum_{i=1}^{n+1-s} (a_i a_{i+s} + b_i b_{i+s}), \quad s = 1, \dots, n, \\ N_C(s) + N_D(s) &= \sum_{i=1}^{n-s} (c_i c_{i+s} + d_i d_{i+s}), \quad s = 1, \dots, n-1. \end{aligned}$$

Then

$$(10) \quad \begin{aligned} N_A(s) + N_B(s) + N_C(s) + N_D(s) &= \sum_{i=1}^{n-s} (a_i a_{i+s} + b_i b_{i+s} + c_i c_{i+s} + d_i d_{i+s}) \\ &\quad + a_{n+1} a_{n+1-s} + b_{n+1} b_{n+1-s} = 0, \\ &\quad s = 1, \dots, n-1, \end{aligned}$$

and since $xy \equiv (x+y-1) \pmod{4}$ whenever $x, y = \pm 1$, we have from (10)

$$\begin{aligned} F(s) &:= \sum_{i=1}^{n-s} (a_i + a_{i+s} + b_i + b_{i+s} + c_i + c_{i+s} + d_i + d_{i+s}) \\ &\quad + a_{n+1} + a_{n+1-s} + b_{n+1} + b_{n+1-s} \equiv 2 \pmod{4}, \quad s = 1, \dots, n-1, \end{aligned}$$

or

$$(11) \quad \begin{aligned} F(s) &= \sum_{i=1}^{n-s} (a_i + b_i + c_i + d_i) + \sum_{i=s+1}^n (a_i + b_i + c_i + d_i) \\ &\quad + a_{n+1} + b_{n+1} + a_{n+1-s} + b_{n+1-s} \\ &\equiv 2 \pmod{4}, \quad s = 1, \dots, n-1. \end{aligned}$$

Now

$$(12) \quad \begin{aligned} F(s-1) - F(s) &= a_s + b_s + c_s + d_s + a_{n+2-s} + b_{n+2-s} + c_{n+1-s} + d_{n+1-s} \\ &\equiv 0 \pmod{4}, \quad s = 2, \dots, n-1, \end{aligned}$$

and for $s = n-1$ we have from (11)

$$(13) \quad a_1 + b_1 + c_1 + d_1 + a_n + b_n + c_n + d_n + a_{n+1} + b_{n+1} + a_2 + b_2 \equiv 2 \pmod{4}.$$

Also, from

$$N_A(n) + N_B(n) = a_1 a_{n+1} + b_1 b_{n+1} = 0$$

we have

$$(14) \quad a_1 + b_1 + a_{n+1} + b_{n+1} \equiv 2 \pmod{4}.$$

From (12)–(14) we obtain

$$(15) \quad (a_s + b_s + a_{n+2-s} + b_{n+2-s}) + (c_s + d_s + c_{n+1-s} + d_{n+1-s}) \\ \equiv 0 \pmod{4}, \quad s = 2, \dots, n.$$

If we set $n + 2 - s$ instead of s in (15), then

$$(16) \quad (a_{n+2-s} + b_{n+2-s} + a_s + b_s) + (c_{n+2-s} + d_{n+2-s} + c_{s-1} + d_{s-1}) \\ \equiv 0 \pmod{4}, \quad s = 2, \dots, n.$$

From (15) and (16) we have

$$(17) \quad c_{s-1} + c_s + d_{s-1} + d_s + c_{n+1-s} + c_{n+2-s} + d_{n+1-s} + d_{n+2-s} \\ \equiv 0 \pmod{4}, \quad s = 2, \dots, n.$$

For n odd, (17) becomes

$$(18) \quad c_s + c_{n+1-s} + d_s + d_{n+1-s} \equiv 0 \pmod{4}, \quad s = 2, \dots, n,$$

because with $s = (n + 1)/2$, (17) gives

$$c_{(n-1)/2} + c_{(n+3)/2} + d_{(n-1)/2} + d_{(n+3)/2} \equiv 0 \pmod{4};$$

then set $s = (n + 1)/2 - 1$, etc.

From (15) and (18) we have

$$(19) \quad a_s + b_s + a_{n+2-s} + b_{n+2-s} \equiv 0 \pmod{4}, \quad s = 2, \dots, n.$$

If we set $n + 1 - s$ instead of s in (15), then we have similarly

$$(20) \quad a_s + a_{s+1} + b_s + b_{s+1} + a_{n+1-s} + b_{n+1-s} + a_{n+2-s} + b_{n+2-s} \\ \equiv 0 \pmod{4}, \quad s = 2, \dots, n - 1.$$

For n even, using the same argument as before, (20) becomes

$$(21) \quad a_s + b_s + a_{n+2-s} + b_{n+2-s} \equiv 0 \pmod{4}, \quad s = 2, \dots, n - 1,$$

and from (15)

$$(22) \quad c_s + d_s + c_{n+1-s} + d_{n+1-s} \equiv 0 \pmod{4}, \quad s = 2, \dots, n - 1.$$

Therefore, (14), (18), (19), (21), and (22) give the required result. \square

In (19) and (21) it is enough to take $s = 2, \dots, [(n + 1)/2]$, and in (18) and (22), $s = 1, \dots, [n/2]$.

Before describing the algorithm, we need the following:

Given the sequence $E = \{e_1, \dots, e_n\}$, we define the m subsequences, for some $m = 2, 3, \dots, n$:

$$E_1 = \{e_1, e_{1+m}, \dots, e_{1+s_1 \cdot m}\} \quad \text{with } s_1 = \left\lceil \frac{n-1}{m} \right\rceil,$$

$$E_2 = \{e_2, e_{2+m}, \dots, e_{2+s_2 \cdot m}\} \quad \text{with } s_2 = \left\lceil \frac{n-2}{m} \right\rceil,$$

$$\vdots$$

$$E_m = \{e_m, e_{2m}, \dots, e_{m+s_m \cdot m}\} \quad \text{with } s_m = \left\lceil \frac{n-m}{m} \right\rceil,$$

or

$$E_i = \{e_i, e_{i+m}, \dots, e_{i+s_i \cdot m}\} \quad \text{with } s_i = \left\lfloor \frac{n-i}{m} \right\rfloor, \quad i = 1, \dots, m,$$

with associated polynomials

$$E_i(z) = \sum_{j=0}^{s_i} z^j \cdot e_{i+j \cdot m}, \quad i = 1, \dots, m.$$

Then

$$E(z) = E_1(z^m) + zE_2(z^m) + \dots + z^{m-1}E_m(z^m)$$

or

$$(23) \quad E(z) = \sum_{i=1}^m z^{i-1} E_i(z^m).$$

Theorem 2. *If $A = \{a_1, \dots, a_{n+1}\}$, $B = \{b_1, \dots, b_{n+1}\}$, $C = \{c_1, \dots, c_n\}$, $D = \{d_1, \dots, d_n\}$ are $(1, -1)$ -sequences of lengths $n + 1$, $n + 1$, n , n , then they are BS($2n + 1$) if and only if for some $m = 2, \dots, n + 1$*

$$\begin{aligned} & \sum_{i=1}^m (A_i(z^m)A_i(z^{-m}) + B_i(z^m)B_i(z^{-m}) \\ & \quad + C_i(z^m)C_i(z^{-m}) + D_i(z^m)D_i(z^{-m})) = 4n + 2, \\ (24) \quad & \sum_{i=1}^{m-s} (A_i(z^m)A_{i+s}(z^{-m}) + B_i(z^m)B_{i+s}(z^{-m}) \\ & \quad + C_i(z^m)C_{i+s}(z^{-m}) + D_i(z^m)D_{i+s}(z^{-m})) \\ & \quad + z^m \sum_{i=1}^s (A_{i+m-s}(z^m)A_i(z^{-m}) + B_{i+m-s}(z^m)B_i(z^{-m}) \\ & \quad \quad + C_{i+m-s}(z^m)C_i(z^{-m}) + D_{i+m-s}(z^m)D_i(z^{-m})) = 0, \\ & \quad \quad \quad s = 1, \dots, [m/2]. \end{aligned}$$

Proof. Writing $A(z)$, $B(z)$, $C(z)$, $D(z)$ as in (23), and equating all coefficients of z^t in (7), where $t \equiv s \pmod m$, we find the above relations (24) for $s = 1, \dots, m - 1$. By setting z^{-1} instead of z in (24) we see that it is enough to take $s = 1, \dots, [m/2]$. \square

For $m = 2$ we have

$$\begin{aligned} (25) \quad & A(z) = A_1(z^2) + zA_2(z^2), \\ & B(z) = B_1(z^2) + zB_2(z^2), \\ & C(z) = C_1(z^2) + zC_2(z^2), \\ & D(z) = D_1(z^2) + zD_2(z^2). \end{aligned}$$

From (24) and (25) we conclude that there are four isomorphic transformations for the $BS(2n + 1)$, A, B, C, D , i.e.,

- (i) interchange A and B and/or C and D ,
- (ii) reverse one or more sequences,
- (iii) negate one or more sequences,
- (iv) negate alternate elements in all sequences.

Corollary 1. *If A, B, C, D are $BS(2n + 1)$ and given $m \in \{2, 3, \dots, n + 1\}$,*

$$(27) \quad \begin{aligned} \text{(i)} \quad k_{im} &= \sum_{j \equiv i \pmod m} a_j, & r_{im} &= \sum_{j \equiv i \pmod m} b_j, \\ p_{im} &= \sum_{j \equiv i \pmod m} c_j, & q_{im} &= \sum_{j \equiv i \pmod m} d_j, \end{aligned}$$

$$(28) \quad \begin{aligned} \text{(ii)} \quad K_m &= \{k_{1m}, \dots, k_{mm}\}, & R_m &= \{r_{1m}, \dots, r_{mm}\}, \\ P_m &= \{p_{1m}, \dots, p_{mm}\}, & Q_m &= \{q_{1m}, \dots, q_{mm}\}, \end{aligned}$$

$$(29) \quad \begin{aligned} \text{(iii)} \quad N_K(s) &= \sum_{i=1}^{m-s} k_{im} k_{i+s, m}, & N_R(s) &= \sum_{i=1}^{m-s} r_{im} r_{i+s, m}, \\ N_P(s) &= \sum_{i=1}^{m-s} p_{im} p_{i+s, m}, & N_Q(s) &= \sum_{i=1}^{m-s} q_{im} q_{i+s, m}, \end{aligned}$$

then for the given $m \in \{2, 3, \dots, n + 1\}$,

$$(30) \quad \begin{aligned} &N_K(0) + N_R(0) + N_P(0) + N_Q(0) \\ &= k_{1m}^2 + \dots + k_{mm}^2 + r_{1m}^2 + \dots + r_{mm}^2 \\ &\quad + p_{1m}^2 + \dots + p_{mm}^2 + q_{1m}^2 + \dots + q_{mm}^2 \\ &= 4n + 2, \end{aligned}$$

$$(31) \quad \begin{aligned} &N_K(s) + N_R(s) + N_P(s) + N_Q(s) + N_K(m - s) + N_R(m - s) \\ &\quad + N_P(m - s) + N_Q(m - s) = 0, \quad s = 1, \dots, [m/2]. \end{aligned}$$

Proof. If we set $z^m = 1$, then $A_i(z^m), B_i(z^m), C_i(z^m), D_i(z^m)$ give

$$A_i(1) = k_{im}, \quad B_i(1) = r_{im}, \quad C_i(1) = p_{im}, \quad D_i(1) = q_{im},$$

and (24) becomes

$$\begin{aligned} &\sum_{i=1}^m (k_{im}^2 + r_{im}^2 + p_{im}^2 + q_{im}^2) = 4n + 2, \\ &\sum_{i=1}^{m-s} (k_{im} k_{i+s, m} + r_{im} r_{i+s, m} + p_{im} p_{i+s, m} + q_{im} q_{i+s, m}) \\ &\quad + \sum_{i=1}^s (k_{im} k_{i+m-s, m} + r_{im} r_{i+m-s, m} + p_{im} p_{i+m-s, m} + q_{im} q_{i+m-s, m}) = 0, \end{aligned}$$

that is,

$$\begin{aligned} N_K(0) + N_R(0) + N_P(0) + N_Q(0) &= 4n + 2, \\ N_K(s) + N_R(s) + N_P(s) + N_Q(s) + N_K(m-s) + N_R(m-s) \\ &+ N_P(m-s) + N_Q(m-s) = 0, \quad s = 1, \dots, [m/2]. \quad \square \end{aligned}$$

Note that

$$(32) \quad \begin{aligned} k_{jm} &= k_{j,2m} + k_{j+m,2m}, & r_{jm} &= r_{j,2m} + r_{j+m,2m}, & j &= 1, \dots, m. \\ p_{jm} &= p_{j,2m} + p_{j+m,2m}, & q_{jm} &= q_{j,2m} + q_{j+m,2m}, \end{aligned}$$

From (9) we can find further restrictions for $k_{im}, r_{im}, p_{im}, q_{im}$, as noted in the following theorem.

Theorem 3. *If A, B, C, D are $BS(2n + 1)$, then for a given $m \in \{2, 3, \dots, n + 1\}$,*

$$(33) \quad \begin{aligned} k_{1m} + r_{1m} + k_{n+1,m} + r_{n+1,m} &\equiv \begin{cases} 2 \pmod 4 & \text{if } n \not\equiv 0 \pmod m, \\ 0 \pmod 4 & \text{if } n \equiv 0 \pmod m, \end{cases} \\ k_{jm} + r_{jm} + k_{n+2-j,m} + r_{n+2-j,m} &\equiv 0 \pmod 4, & j &= 2, \dots, m, \\ p_{jm} + q_{jm} + p_{n+1-j,m} + q_{n+1-j,m} &\equiv 0 \pmod 4, & j &= 1, \dots, m, \end{aligned}$$

where $k_{im}, r_{im}, p_{im}, q_{im}$ are defined in (27).

Proof. Summing all relations in (9) with $s \equiv j \pmod m$, we obtain the above result. \square

The Algorithm. Since it is difficult to find directly the values of $a_1, \dots, a_{n+1}, b_1, \dots, b_{n+1}, c_1, \dots, c_n, d_1, \dots, d_n$, we find the values of

$$k_{1m}, \dots, k_{mm}, r_{1m}, \dots, r_{mm}, p_{1m}, \dots, p_{mm}, q_{1m}, \dots, q_{mm}$$

as defined in (27). Our algorithm relies on Corollary 1. To avoid calculating isomorphic $BS(2n + 1)$, on the basis of the four properties given in (26), we can always take

$$(34) \quad k_{11} \geq r_{11} \geq 0, \quad p_{11} \geq q_{11} \geq 0,$$

where $k_{11}, r_{11}, p_{11}, q_{11}$ is the sum of the elements of A, B, C, D , respectively. If n is odd, then $k_{12} \geq k_{22} \geq 0, r_{12} \geq r_{22}$. If n is even, then $p_{12} \geq p_{22} \geq 0, q_{12} \geq q_{22}$.

Step 1. Find all $k_{11}, r_{11}, p_{11}, q_{11}$ satisfying

$$k_{11} \geq r_{11} \geq 0, \quad p_{11} \geq q_{11} \geq 0, \quad k_{11}^2 + r_{11}^2 + p_{11}^2 + q_{11}^2 = 4n + 2,$$

and for

- (i) n even, take k_{11}, r_{11} odd,
 p_{11}, q_{11} even,
- (ii) n odd, take k_{11}, r_{11} even,
 p_{11}, q_{11} odd.

Step 2. For every quadruple $k_{11}, r_{11}, p_{11}, q_{11}$ and given $m \in \{2, 3, \dots, n+1\}$, find $k_{1m}, \dots, k_{mm}, r_{1m}, \dots, r_{mm}, p_{1m}, \dots, p_{mm}, q_{1m}, \dots, q_{mm}$ satisfying

- (i) $k_{11} = k_{1m} + \dots + k_{mm}, r_{11} = r_{1m} + \dots + r_{mm}, p_{11} = p_{1m} + \dots + p_{mm}, q_{11} = q_{1m} + \dots + q_{mm},$
- (ii) k_{jm}, r_{jm} are both odd (even) if $[(n+1-j)/m] + 1$ is odd (even) and

$$|k_{jm}| \leq \left\lceil \frac{n+1-j}{m} \right\rceil + 1, \quad |r_{jm}| \leq \left\lceil \frac{n+1-j}{m} \right\rceil + 1, \quad j = 1, \dots, m,$$

p_{jm}, q_{jm} are both odd (even) if $[(n-j)/m] + 1$ is odd (even) and

$$|p_{jm}| \leq \left\lceil \frac{n-j}{m} \right\rceil + 1, \quad |q_{jm}| \leq \left\lceil \frac{n-j}{m} \right\rceil + 1, \quad j = 1, \dots, m,$$

(iii)

$$k_{1m} + r_{1m} + k_{n+1,m} + r_{n+1,m} \equiv \begin{cases} 2 \pmod 4 & \text{if } n \not\equiv 0 \pmod m, \\ 0 \pmod 4 & \text{if } n \equiv 0 \pmod m, \end{cases}$$

$$k_{jm} + r_{jm} + k_{n+2-j,m} + r_{n+2-j,m} \equiv 0 \pmod 4, \quad j = 2, \dots, m,$$

$$p_{jm} + q_{jm} + p_{n+1-j,m} + q_{n+1-j,m} \equiv 0 \pmod 4, \quad j = 1, \dots, m,$$

(iv) $k_{1m}^2 + \dots + k_{mm}^2 + r_{1m}^2 + \dots + r_{mm}^2 + p_{1m}^2 + \dots + p_{mm}^2 + q_{1m}^2 + \dots + q_{mm}^2 = 4n+2,$

(v) $N_K(s) + N_K(m-s) + N_R(s) + N_R(m-s) + N_P(s) + N_P(m-s) + N_Q(s) + N_Q(m-s) = 0, s = 1, \dots, [m/2],$ where

$$N_K(s) = \sum_{j=1}^{m-s} k_{jm} k_{j+s,m}, \quad N_R(s) = \sum_{j=1}^{m-s} r_{jm} r_{j+s,m},$$

$$N_P(s) = \sum_{j=1}^{m-s} p_{jm} p_{j+s,m}, \quad N_Q(s) = \sum_{j=1}^{m-s} q_{jm} q_{j+s,m}.$$

Step 3. (i) For every $k_{1m}, \dots, k_{mm}, r_{1m}, \dots, r_{mm}, p_{1m}, \dots, p_{mm}, q_{1m}, \dots, q_{mm}$ found in Step 2, find $k_{1,2m}, \dots, k_{2m,2m}, r_{1,2m}, \dots, r_{2m,2m}, p_{1,2m}, \dots, p_{2m,2m}, q_{1,2m}, \dots, q_{2m,2m}$ satisfying

$$\begin{aligned} k_{jm} &= k_{j,2m} + k_{j+m,2m}, & r_{jm} &= r_{j,2m} + r_{j+m,2m}, & j &= 1, \dots, m. \\ p_{jm} &= p_{j,2m} + p_{j+m,2m}, & q_{jm} &= q_{j,2m} + q_{j+m,2m}, \end{aligned}$$

(ii) Go to Step 2(ii)-(v), setting $2m$ instead of m .

Step 4. Stop when $m \geq n+1$ and examine if

$$N_K(s) + N_R(s) + N_P(s) + N_Q(s) = 0, \quad s = 1, \dots, m-1,$$

because for $m \geq n+1,$

$$k_{jm} = 0, 1, -1, \quad r_{jm} = 0, 1, -1,$$

$$p_{jm} = 0, 1, -1, \quad q_{jm} = 0, 1, -1.$$

TABLE 1

Base sequences $BS(2n+1) A, B, C, D$ of lengths $n+1, n+1, n, n$ or Suitable sequences $SS(2n+1)(A+B)/2, (A-B)/2, (C+D)/2, (C-D)/2$

Length	Sum of Squares for Suitable Sequences	Sequence
$n = 19$	$2n + 1 = 39 = 6^2 + 1^2 + 1^2 + 1^2$	$\left\{ \begin{array}{l} - + - + - + - + - + - + - + - + - + - + - + - + - \\ + - - + - + - + - + - + - + - + - + - + - + - \\ + - + + + + - + - + - + - + - + - + - + - + - + - \\ + + + + + - + - + - + - + - + - + - + - + - + - \end{array} \right\}$
$n = 19$	$2n + 1 = 39 = 5^2 + 3^2 + 2^2 + 1^2$	$\left\{ \begin{array}{l} + + + - - + - + - + - + - + - + - + - + - + - \\ + - - + - + - + - + - + - + - + - + - + - + - \\ + + + - - + - + - + - + - + - + - + - + - + - \\ + + - - + - + - + - + - + - + - + - + - + - + - \end{array} \right\}$
$n = 20$	$2n + 1 = 41 = 6^2 + 2^2 + 1^2 + 0^2$	see [1], or $\left\{ \begin{array}{l} + + - + - + - + - + - + - + - + - + - + - + - \\ - - + + - + - + - + - + - + - + - + - + - + - \\ + - - + - + - + - + - + - + - + - + - + - + - \\ + + + + - + - + - + - + - + - + - + - + - + - \end{array} \right\}$
$n = 20$	$2n + 1 = 41 = 5^2 + 4^2 + 0^2 + 0^2$	see [7], or $\left\{ \begin{array}{l} - - + + + - - + - + - + - + - + - + - + - + - \\ + - + + + + - + - + - + - + - + - + - + - + - \\ - + + - + - + - + - + - + - + - + - + - + - + - \\ + - - + - + - + - + - + - + - + - + - + - + - \end{array} \right\}$
$n = 20$	$2n + 1 = 41 = 4^2 + 4^2 + 3^2 + 0^2$	$\left\{ \begin{array}{l} - - + - + - + - + - + - + - + - + - + - + - + \\ + - - + - + - + - + - + - + - + - + - + - + - \\ + - - + - + - + - + - + - + - + - + - + - + - \\ + - + + + - + - + - + - + - + - + - + - + - + - \end{array} \right\}$
$n = 21$	$2n + 1 = 43 = 5^2 + 4^2 + 1^2 + 1^2$	$\left\{ \begin{array}{l} - + + + - + - + - + - + - + - + - + - + - + - \\ + + + + + - + - + - + - + - + - + - + - + - + - \\ - - - + + + - + - + - + - + - + - + - + - + - \\ + - + + + - + - + - + - + - + - + - + - + - + - \end{array} \right\}$
$n = 21$	$2n + 1 = 43 = 5^2 + 3^2 + 3^2 + 0^2$	$\left\{ \begin{array}{l} + + + + - - + - - + - - + - - + - - + - - + - \\ + + - + + + - + - + - + - + - + - + - + - + - \\ - - - + + + - + - + - + - + - + - + - + - + - \\ + - - + + + + - + - + - + - + - + - + - + - + - \end{array} \right\}$
$n = 21$	$2n + 1 = 43 = 4^2 + 3^2 + 3^2 + 3^2$	see [7], or $\left\{ \begin{array}{l} - - + - + - - + + + + - - + - + - + - + - + - \\ - - + + - + - + - + - + - + - + - + - + - + - \\ - - - + + + - + - + - + - + - + - + - + - + - \\ + - + + + - + - + - + - + - + - + - + - + - + - \end{array} \right\}$
$n = 22$	$2n + 1 = 45 = 6^2 + 3^2 + 0^2 + 0^2$	$\left\{ \begin{array}{l} + - + - - + - - + - + - + - + - + - + - + - + \\ - + + - - + + + - + + + - + + + - + + + - + + + \\ + + - + - + - + - + - + - + - + - + - + - + - \\ - + - + + + - + - + - + - + - + - + - + - + - \end{array} \right\}$
$n = 22$	$2n + 1 = 45 = 6^2 + 2^2 + 2^2 + 1^2$	$\left\{ \begin{array}{l} - - + - + + + - - + - + - + - + - + - + - + - \\ + - - + + + - - + - + - + - + - + - + - + - + \\ + + - + - + - + - + - + - + - + - + - + - + - \\ + + + + - + - + - + - + - + - + - + - + - + - \end{array} \right\}$
$n = 22$	$2n + 1 = 45 = 5^2 + 4^2 + 2^2 + 0^2$	$\left\{ \begin{array}{l} - + - + - + - + - + - + - + - + - + - + - + - \\ + - + - + - + - + - + - + - + - + - + - + - + \\ + - + + + - + - + - + - + - + - + - + - + - + \\ + - + + - + - + - + - + - + - + - + - + - + - \end{array} \right\}$
$n = 22$	$2n + 1 = 45 = 4^2 + 4^2 + 3^2 + 2^2$	see [7], or $\left\{ \begin{array}{l} + - - + + - + - + - + - + - + - + - + - + - + \\ - + - - + - + - + - + - + - + - + - + - + - + \\ + + + - + - + - + - + - + - + - + - + - + - + \\ + - + + + + - + - + - + - + - + - + - + - + - \end{array} \right\}$
$n = 23$	$2n + 1 = 47 = 6^2 + 3^2 + 1^2 + 1^2$	$\left\{ \begin{array}{l} + + + - - + - - + - + - + - + - + - + - + - + \\ + + - + - + - + - + - + - + - + - + - + - + - \\ + + + - + - + - + - + - + - + - + - + - + - + \\ + - + - - + - + - + - + - + - + - + - + - + - \end{array} \right\}$
$n = 23$	$2n + 1 = 47 = 5^2 + 3^2 + 3^2 + 2^2$	see [1], or $\left\{ \begin{array}{l} - + + + - + - + - + - + - + - + - + - + - + - \\ - + + + + - + - + - + - + - + - + - + - + - + \\ + + - + - + - + - + - + - + - + - + - + - + \\ - - - + + + - + - + - + - + - + - + - + - + - \end{array} \right\}$

TABLE 1 (continued)

Length	Sum of Squares for Suitable Sequences	Sequence
$n = 24$	$2n + 1 = 49 = 7^2 + 0^2 + 0^2 + 0^2$	$\left\{ \begin{array}{l} - - - - + + + - + + - + + + + + + + - + - + - + \\ + - - + + + + + + + - - - + + - + - + + + - + \\ - + - + + + + + - + - + - + - + - + + + - - - \\ + + - + - - - + - + - - - + + + - - + + - - - \end{array} \right\}$
$n = 24$	$2n + 1 = 49 = 6^2 + 3^2 + 2^2 + 0^2$	$\left\{ \begin{array}{l} - + + + - - - - - + + - + + + + + - + - + - - + \\ + - + - + - - - + + + - - + + + - + + - + + \\ + + + + - + + - + + + + - - - + - + - + - + - \\ + + - + + - - + + + + + - - + - + - + + + + \end{array} \right\}$
$n = 24$	$2n + 1 = 49 = 5^2 + 4^2 + 2^2 + 0^2$	$\left\{ \begin{array}{l} - + - + - + + + + + - - - - + + - + - + - - + + \\ + + + + + - + - + - + - + - + - + - + + + + \\ - + + + - - + + + + + - + + - + - + - + + + \\ + + + + - - + + - + + + - + + - + - + - - - \end{array} \right\}$
$n = 24$	$2n + 1 = 49 = 4^2 + 4^2 + 4^2 + 1^2$	$\left\{ \begin{array}{l} + + + + - + + - - + - - - + - + + - + - + + - \\ + - + - + + + - - + + + + + - - - - + + + \\ - + + + - + + + + + - - + - - + - + - - - \\ + - + + - + + + + + - - - - + - + - - - - \\ + + - - + - + - + - - + + + - + + + - + + + \end{array} \right\}$

Remark. C. H. Yang and J. Yang (private communication) have constructed the equivalent of the sequences shown in Table 1 by constructing base, normal, or near normal sequences.

3. TURYN SEQUENCES

Our algorithm requires too much computer time for the construction of $BS(2n+1)$ for $n \geq 25$. It is desirable to construct $BS(4n+3)$ from $BS(2n+1)$.

Theorem 4. *If A, B, C, D are $BS(2n+1)$, and their associated polynomials satisfy*

$$(35) \quad A(z)C(z^{-1}) + zC(z)A(z^{-1}) = 0, \quad z \neq 0,$$

then the sequences

$$(36) \quad X = \{1, A|C\}, \quad Y = \{-1, A|C\}, \quad Z = \{B|D\}, \quad W = \{B|-D\}$$

are $BS(4n+3)$, where

$$\begin{aligned} A|C &= \{a_1, c_1, a_2, c_2, \dots, a_n, c_n, a_{n+1}\}, \\ B|D &= \{b_1, d_1, b_2, d_2, \dots, b_n, d_n, b_{n+1}\}. \end{aligned}$$

Proof. The polynomials associated with X, Y, Z, W are

$$\begin{aligned} X(z) &= 1 + z(A(z^2) + zC(z^2)), & Y(z) &= -1 + z(A(z^2) + zC(z^2)), \\ Z(z) &= B(z^2) + zD(z^2), & W(z) &= B(z^2) - zD(z^2). \end{aligned}$$

Then

$$\begin{aligned} &X(z)X(z^{-1}) + Y(z)Y(z^{-1}) + Z(z)Z(z^{-1}) + W(z)W(z^{-1}) \\ &= 2 + 2(A(z^2)A(z^{-2}) + B(z^2)B(z^{-2}) + C(z^2)C(z^{-2}) + D(z^2)D(z^{-2})) \\ &\quad + 2z^{-1}(z^2C(z^2)A(z^{-2}) + A(z^2)C(z^{-2})) = 8n + 6, \quad z \neq 0. \quad \square \end{aligned}$$

If we define the cross correlations

$$(37) \quad N_{AC}(s) = \sum_{i=1}^{n-s} a_i c_{i+s}, \quad N_{CA}(s) = \sum_{i=1}^{n+1-s} c_i a_{i+s}, \quad s = 0, 1, \dots, n,$$

then

$$C(z)A(z^{-1}) = \sum_{s=0}^n N_{AC}(s)z^s + \sum_{s=1}^n N_{CA}(s)z^{-s}$$

and

$$zC(z)A(z^{-1}) + A(z)C(z^{-1}) = \sum_{s=1}^n (N_{AC}(s-1) + N_{CA}(s))(z^s + z^{-s+1}).$$

Therefore, $zC(z)A(z^{-1}) + A(z)C(z^{-1}) = 0$ is equivalent to

$$(38) \quad N_{AC}(s-1) + N_{CA}(s) = 0, \quad s = 1, \dots, n.$$

Definition. If the sequences A, B, C, D are $BS(2n+1)$ and satisfy

$$N_{AC}(s-1) + N_{CA}(s) = 0, \quad s = 1, \dots, n,$$

then they are called *Turyn sequences* (abbreviated as $TS(2n+1)$).

Theorem 5 (see also [3, pp. 139–142]). *If A, B, C, D are $TS(2n+1)$, then*

(i) *For n odd,*

$$A = \{A_1, -A_1^*\}, \quad B = \{b_1, B_1, -B_1^*, b_1\}, \\ C = \{C_1, c_{(n+1)/2}, C_1^*\}, \quad D = \{D_1, d_{(n+1)/2}, D_1^*\},$$

where

$$A_1 = \{a_1, \dots, a_{(n+1)/2}\}, \quad B_1 = \{b_2, \dots, b_{(n+1)/2}\}, \\ C_1 = \{c_1, \dots, c_{(n-1)/2}\}, \quad D_1 = \{d_1, \dots, d_{(n-1)/2}\}.$$

(ii) *For n even,*

$$A = \{A_1, a_{n/2+1}, A_1^*\}, \quad B = \{b_1, B_1, b_{n/2+1}, B_1^*, -b_1\}, \\ C = \{C_1, -C_1^*\}, \quad D = \{D_1, -D_1^*\},$$

where

$$A_1 = \{a_1, \dots, a_{n/2}\}, \quad B_1 = \{b_2, \dots, b_{n/2}\}, \\ C_1 = \{c_1, \dots, c_{n/2}\}, \quad D_1 = \{d_1, \dots, d_{n/2}\}.$$

Proof. From (38) and from $x \cdot y \equiv (x + y - 1) \pmod 4$ when $x, y = \pm 1$, we have

$$F(s) := \sum_{i=1}^{n+1-s} a_i + \sum_{i=s+1}^{n+1} a_i + \sum_{i=1}^{n+1-s} c_i + \sum_{i=s}^n c_i \equiv 2(n+1-s) \pmod 4, \quad s = 1, \dots, n;$$

for $s = n + 1$, (37) gives

$$a_1 + a_{n+1} + c_1 + c_n \equiv 2 \pmod{4}.$$

Then

$$(39) \quad F(s-1) - F(s) = a_{n+2-s} + a_s + c_{n+2-s} + c_{s-1} \equiv 2 \pmod{4}, \\ s = 2, \dots, n+1.$$

Proceeding as in Theorem 1, we obtain from (39) and (9)

(i) For n odd:

$$\begin{aligned} a_s + a_{n+2-s} &\equiv 0 \pmod{4}, & s = 1, \dots, n+1, \\ c_s + c_{n+1-s} &\equiv 2 \pmod{4}, & s = 1, \dots, n, \\ b_s + b_{n+2-s} &\equiv \begin{cases} 2 \pmod{4}, & s = 1, n+1, \\ 0 \pmod{4}, & s = 2, \dots, n. \end{cases} \end{aligned}$$

(ii) For n even:

$$\begin{aligned} a_s + a_{n+2-s} &\equiv 2 \pmod{4}, & s = 1, \dots, n+1, \\ b_s + b_{n+2-s} &\equiv \begin{cases} 0 \pmod{4}, & s = 1, n+1, \\ 2 \pmod{4}, & s = 2, \dots, n, \end{cases} \\ c_s + c_{n+1-s} &\equiv 0 \pmod{4}, & s = 1, \dots, n, \\ d_s + d_{n+1-s} &\equiv 0 \pmod{4}, & s = 1, \dots, n, \end{aligned}$$

which proves Theorem 5. \square

Our algorithm in §2 can now be modified to give $\text{TS}(2n+1)$. From Theorem 5 we have:

(i) For n odd:

$$\begin{aligned} k_{jm} &= -k_{n+2-j, m}, & j = 1, \dots, m, \\ r_{1m} + r_{n+1, m} &= 2, \\ r_{jm} &= -r_{n+2-j, m}, & j = 2, \dots, m, \\ p_{jm} &= p_{n+1-j, m}, & j = 1, \dots, m, \\ q_{jm} &= q_{n+1-j, m}, & j = 1, \dots, m. \end{aligned}$$

(ii) For n even:

$$\begin{aligned} k_{jm} &= k_{n+2-j, m}, & j = 1, \dots, m, \\ r_{jm} &= r_{n+2-j, m}, & j = 2, \dots, m, \\ r_{1m} - r_{n+1, m} &= 2 & \text{for } n \not\equiv 0 \pmod{4}, \\ p_{jm} &= -p_{n+1-j, m}, & j = 1, \dots, m, \\ q_{jm} &= -q_{n+1-j, m}, & j = 1, \dots, m. \end{aligned}$$

It is known that $TS(2n + 1)$ exists for $n \leq 7$ and $n = 12, 14$; they cannot exist for $n = 10, 11, 16, 17$, and for $n = 8, 9, 13, 15$, $TM(2n + 1)$ might exist but an exhaustive machine search showed that they do not exist (see [1; 3, pp. 142–143]).

In this paper, applying our algorithm, we have done an exhaustive search, and we showed that $TS(2n + 1)$ does not exist for $n = 18, 19, \dots, 27$; of course, for $n = 28, 29$, $TS(2n + 1)$ cannot exist.

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