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MATHEMATICS

EQUILATERAL POINT SETS IN ELLIPTIC GEOMETRY

BY

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1. *Introduction on geometry*

Elliptic space of $r-1$ dimensions E_{r-1} is obtained from r-dimensional vector space R_r with inner product (a, b) as follows. For $1 \leqslant k \leqslant r$, call any k-dimensional linear subspace R_k of R_r a $(k-1)$-dimensional elliptic subspace E_{k-1}, and, for any pair of elliptic points E_0: $x=\lambda a$ and E_0': $x=\mu b$, define the elliptic distance $\delta(E_0, E_0')$ by

$$\cos \delta(E_0, E_0') = \frac{|(a, b)|}{\sqrt{(a, a)(b, b)}}, \qquad 0 \leqslant \delta \leqslant \tfrac{1}{2}\pi,$$

which, by taking $|a|=|b|=1$, reduces to

$$\varepsilon \cos \delta(E_0, E_0') = (a, b), \; \varepsilon = \pm 1, \qquad 0 \leqslant \delta \leqslant \tfrac{1}{2}\pi.$$

The unit sphere in R_r provides a model for E_{r-1}, any elliptic point being represented by a pair of antipodal points. In this model we easily verify, by consideration of the 12 vertices of a regular icosahedron, that the elliptic plane E_2 contains an equilateral 6-tuple all of whose distances equal $\arccos(1/\sqrt{5})$. This leads to the following problem.

1.1 Problem. For all positive integers r, find the integers n such that in E_{r-1} there exists an equilateral n-tuple. What is the distance and the structure of such n-tuples? For any given r, what is the maximum, $n(r)$ say, of n?

The significance of this problem is caused by certain peculiarities in elliptic geometry. For instance, in the elliptic plane two triples with equal elliptic distances need not be superposable by elliptic motion. The existence of equilateral 6-tuples has consequences for the congruence order of the elliptic plane. This congruence order is defined as the smallest integer k with the property that any semimetric space is imbeddable in the elliptic plane whenever each of its k-tuples is. HAANTJES and SEIDEL ([9], [15]) proved that this integer equals 7. The corresponding question for elliptic space of dimensions $\geqslant 3$ is still open, see BLUMENTHAL ([5], p. 269).

In addition, problem 1.1 is related to the theory of polytopes. Let $a_1, \ldots, a_n$ be unit vectors in R_r carrying an equilateral n-tuple of E_{r-1}. The convex hull in R_r of the vectors $\pm a_1, \ldots, \pm a_n$ is a spherical polytope of some kind of regularity, cf. COXETER [7], FEJES TÓTH [8]. The convex hull in R_r of the vectors $\pm a_1 \pm a_2 \ldots \pm a_n$ is an isozonohedron whose 2-faces are congruent rhombs, cf. BILINSKI [3]. Finally we mention that the problem of the biologist TAMMES (cf. [8], p. 214), asking how to select on the sphere n points such that the smallest of their distances is as big as possible, has an analogue in elliptic geometry which was recently attacked by FEJES TÓTH [17].

Problem 1.1 was solved by HAANTJES [10] for $r=3$ and $r=4$; he showed that $n(3)=n(4)=6$. In the present paper the complete solution for $r=5$ is given. It is shown that $n(5)=10$, $n(6)=16$, and $n(7)\geqslant 28$. Thus a conjecture of BLUMENTHAL and KELLY ([4], p. 104) is disproved. The maximum 10-tuple in E_4 shares with the plane equilateral 6-tuple the property that the only eigenvalues of its Gram matrix are 0 and 2, with equal multiplicities. Point sets with this property are shown to exist for infinitely many, but not for all, odd r. However, the lower bounds for $n(r)$ obtained by considering these point sets are often improved by the existence of other equilateral point sets, as for instance the maximum 16-tuple in E_5 and a 28-tuple in E_6. These bounds are treated in section 6; for geometrical comments we refer to section 7.

2. *Introduction on matrices*

In section 3 we shall show that the query for equilateral point sets in elliptic geometry leads to the search for matrices B of order n and elements

$$b_{ii}=0,\ b_{ij}=b_{ji}=\pm 1, \qquad (i\neq j;\ i,j=1,\ldots,n),$$

whose smallest eigenvalue has a high multiplicity. We first give two examples of such matrices. Let I_k denote [1]) the $k\times k$ unit matrix, J_k the $k\times k$ matrix consisting solely of $1'$s, and e_k the $k\times 1$ matrix consisting solely of $1'$s.

2.1 Example. The 16×16 compound matrix consisting of the blocks J_4-I_4 on the diagonal and $2I_4-J_4$ elsewhere has six eigenvalues 5 and ten eigenvalues -3.

2.2 Example. The matrix

$$\begin{pmatrix} 0 & e_9' \\ e_9 & D \end{pmatrix},$$

where D consists of the blocks J_3-I_3 on the diagonal and $2I_3-J_3$ elsewhere, is a 10×10 matrix with five eigenvalues 3 and five eigenvalues -3.

Example 2.1 is representative of some special matrices, to be treated in section 6, which provide bounds for $n(r)$. Example 2.2 provides the solution for $n=10$ to the following problem.

1) We shall delete the subscripts if there is no fear of confusion.

2.3 Problem. Find all symmetric matrices C with diagonal elements 0 and other elements 1 or -1 whose square is a multiple of the unit matrix.

Such C-matrices appear in the literature at various places. Their existence for order n satisfying:

2.4 Condition. $n \equiv 2 \pmod 4$, $n = p^\alpha + 1$, p prime,
was proved by PALEY [11], who used them for the construction of Hadamard matrices of order $2n$. In section 5 we shall see that the search for C-matrices of order n is a block-design problem with $\{v, b, k, r, \lambda\} = \{n, 2n-2, \frac{1}{2}n, n-1, \frac{1}{2}n-1\}$.

RAGHAVARAO ([13], [14]) encountered C-matrices in the construction of weighing designs. He showed that a necessary condition for the existence of rational square matrices of order $n \equiv 2 \pmod 4$ is that $(n-1, -1)_p = 1$ for all primes p, where $(a, b)_p$ denotes the Hilbert normresidue symbol. It can be seen that this implies the nonexistence of C-matrices of order n satisfying:

2.5 Condition. $n \equiv 2 \pmod 4$, $n-1 \neq a^2 + b^2$; a and b integers.

BELEVITCH ([1], [2]) encountered C-matrices in conference telephony. Recently he obtained 2.4 and 2.5, the last condition appearing already in his 1950 paper. His proof of the Bruck–Ryser-type theorem which implies 2.5 led the authors, who independently came to 2.4 and to the impossibility for $n = 22$, to the elementary proof of theorem 5.2.

3. *Re-wording of the problem* [1]

For n elliptic points $A_1, A_2, \ldots, A_n$, carried by the unit vectors $a_1, a_2, \ldots, a_n$ and spanning elliptic space E_{r-1}, the Gram matrix with elements

$$(a_i, a_j) = \varepsilon_{ij} \cos \delta(A_i, A_j),\ \varepsilon_{ij} = \varepsilon_{ji} = \pm 1,\ \varepsilon_{ii} = 1,\ 0 \leqslant \delta \leqslant \tfrac{1}{2}\pi,$$

is symmetric, semipositive definite, and of rank r. Conversely, BLUMENTHAL ([5], p. 208) showed that to any matrix with these properties there exist n points in E_{r-1} whose distances are given by this formula. Thus, in order to investigate equilateral n-tuples in E_{r-1} we ask for symmetric matrices A with elements

$$a_{ii} = 1,\ a_{ij} = \varepsilon_{ij} a, \qquad (i \neq j;\ i, j = 1, \ldots, n)$$

that are semipositive definite with rank r, i.e. that have smallest eigenvalue 0 with multiplicity $n-r$. In other words, we ask for matrices $B = a^{-1}(A - I_n)$ whose smallest eigenvalue has multiplicity $n-r$. Any such matrix B with smallest eigenvalue λ_0 leads to an equilateral n-tuple in

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[1]) The authors thank G. W. Veltkamp for valuable discussions on this section.

E_{r-1} with distance $\delta = \arccos(-1/\lambda_0)$. However, equilateral n-tuples are not characterized by single matrices but by the classes of such matrices under the equivalence relation generated by the following operations.

3.1 Operation. Multiplication by -1 of any row and the corresponding column (replacement of a vector by its opposite has no effect on the corresponding elliptic point).

3.2 Operation. Interchange of two rows and, simultaneously, of the corresponding columns (the order of the points is irrelevant).

Thus we arrive at the following formulation, in terms of matrices, of the original problem.

3.3 Problem. For all positive integers r, find the integers n and the classes under 3.1 and 3.2 of matrices B with order n and elements

$$b_{ii}=0,\ b_{ij}=b_{ji}=\varepsilon_{ij}=\pm 1, \qquad (i\neq j;\ i,j=1,\ldots,n)$$

whose smallest eigenvalue has multiplicity $n-r$. What is this smallest eigenvalue? For any given r, what is the maximum $n(r)$?

Omitting for a moment the condition on the smallest eigenvalue we first try to obtain, for all n, a survey of all equivalence classes, under 3.1 and 3.2, of matrices B. This step may be formulated in combinatorial terms by representing any matrix B of order n by an n-graph, viz. by an n-tuple of points connecting any pair of points i and j if and only if $b_{ij}=-1$. By 3.2 we need not consider an ordering in the n-tuple. The operation corresponding to 3.1 is called complementation and reads as follows.

3.4 Operation. Cancel for any point the existing connections and add for that point the nonexisting connections.

The combinatorial problem is then:

3.5 Problem. Give, for all n, a survey of the equivalence classes, under complementation, of all n-graphs.

4. *Tables*

The number of equivalence classes of n-graphs increases very rapidly with n. We constructed all classes for $n\leqslant 7$.

4.1 Table. The first table contains the classes for $n=2, 3, 4, 5, 6$, each given by one representative. They are arranged according to the partial order of inclusion. Any class, represented by some n-graph, is said to include each of the classes that contains a sub m-graph, $m\leqslant n$, of that n-graph. The number of inclusions is indicated with each class. Furthermore, for each class the eigenvalues, approximate

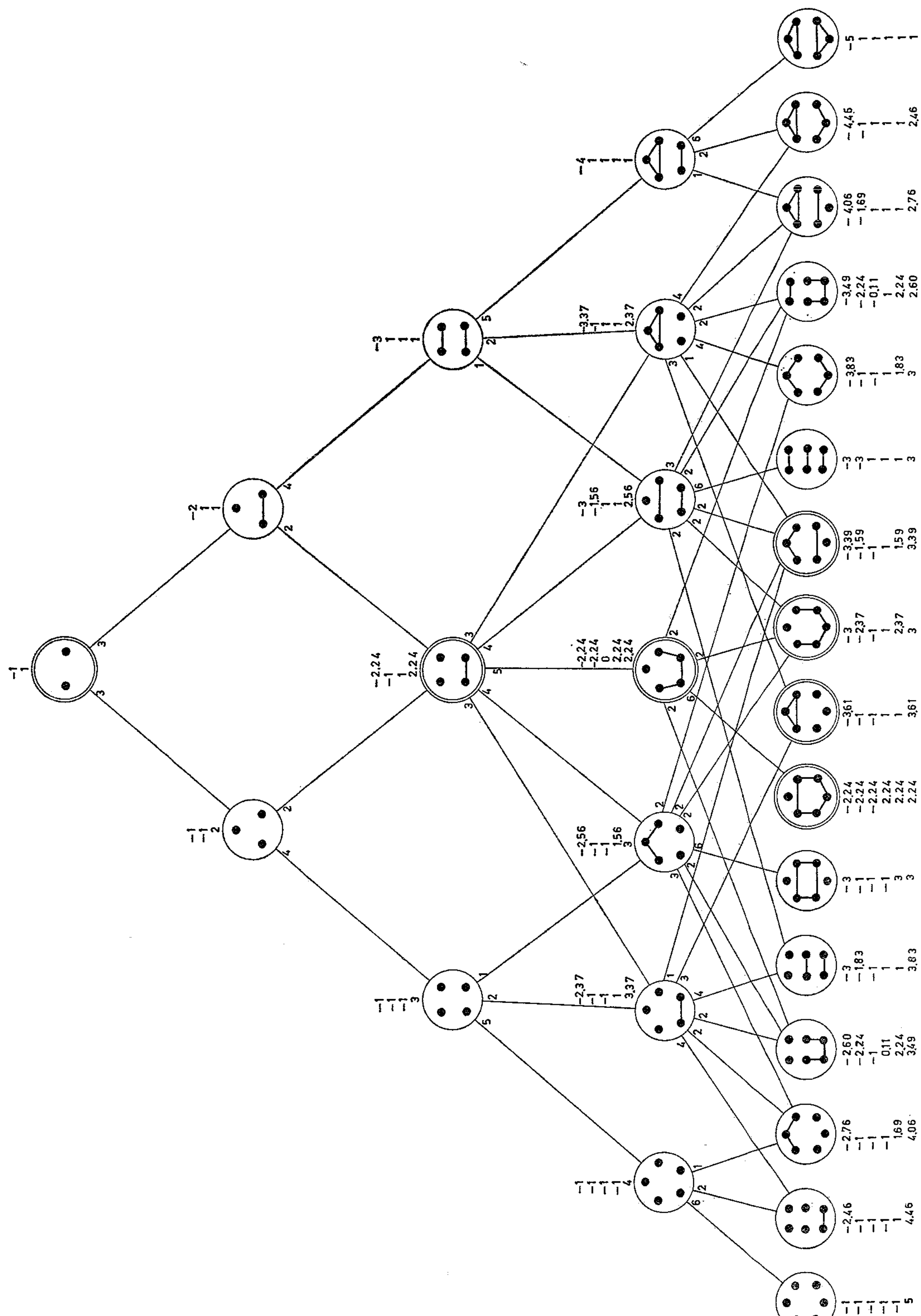
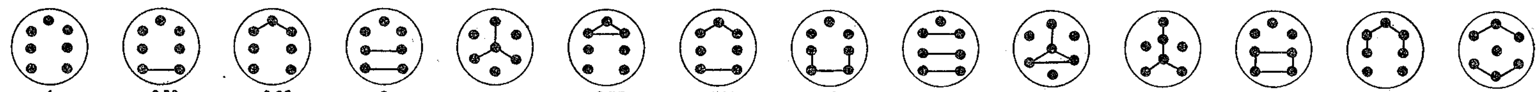
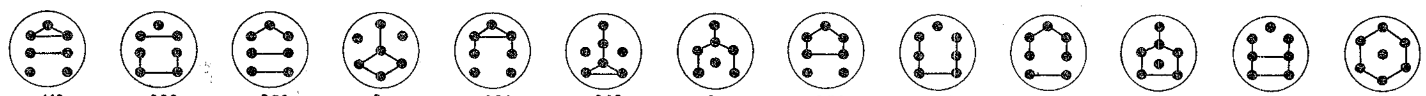


TABLE 4.1



Type 1	2	3	4	5	6	7	8	9	10	11	12	13	14
−1	−2,53	−2,90	−3	−3	−3,77	−3,40	−2,83	−3	−3,64	−2,78	−3,27	−3	−3,83
−1	−1	−1	−2	−1	−1	−1,88	−2,24	−3	−1,60	−2,46	−1	−2,59	−1,63
−1	−1	−1	−1	−1	−1	−1	−1	−1,37	−1	−1	−1	−1	−1
−1	−1	−1	−1	−1	−1	−1	−1	1	−1	−1	−1	−1	−1
−1	−1	−1	1	−1	1	1	0,15	1	1	0,29	−1	1	1,48
−1	1	1,74	1	2	1	170	2,24	1	1,76	2,49	3	2,40	1,83
6	5,53	5,16	5	5	4,77	4,58	4,68	4,37	4,48	4,46	4,27	4,19	4,14



15	16	17	18	19	20	21	22	23	24	25	26	27
−4,12	−3,52	−3,50	−3	−3,39	−3,63	−3	−2,70	−3,49	−3,53	−3	−2,60	−3
−2	−2,24	−3	−2,30	−2,24	−2,54	−2,49	−2,24	−2,44	−3	−2,24	−2,60	−3
−1	−1,58	−1	−1	−1	−1	−1,83	−2,24	−1,51	−1,22	−2,24	−2	−2
1	0,09	1	−1	−1	−0,02	−0,29	−1	−0,11	1	−0,56	0,11	1
1	1	1	0,21	1,29	1	1	2,24	1,36	1	2,24	0,11	1
1	2,24	1,62	3	2,24	2,33	2,78	2,24	2,60	2,32	2,24	3,49	3
4,12	4,01	3,89	4,08	4,10	3,86	3,83	3,70	3,60	3,43	3,56	3,49	3

TABLE 4.2

to two decimals, are given. Two classes are called complementary if they have complementary representative graphs. The selfcomplementary classes are indicated.

4.2 Table. For $n=7$ there are 54 classes, complementary in pairs, half of which are tabulated in the second table. There are no selfcomplementary classes for $n=7$ as can be seen from the eigenvalues.

Some of the 7-graphs were extended to 8-graphs in all possible ways in order to investigate whether the multiplicity of the smallest eigenvalue increases. It turned out that type 26 and its complement are not, and that types 9, 27 and its complement are extendable to 8-graphs with threefold smallest eigenvalue. Type 9 leads to the ladder 8-graph, consisting of four pairs of connected points, which is not extendable in this way any further. Type 27 and its complement lead to an 8-graph which is extendable to the Petersen 10-graph ([12], p. 194) which has eigenvalues -3 and 3, each fivefold.

5. *C-matrices*

B-matrices of order $n=2r$ that have only two distinct eigenvalues with equal multiplicities r are called C-matrices. They are orthogonal with eigenvalues $\sqrt{n-1}$ and $-\sqrt{n-1}$. The orthogonality of any three rows implies the necessary condition $n \equiv 2 \pmod 4$. The following construction was given by Paley [11] and Williamson [16].

5.1 Construction. For C-matrices of order $n=p^\alpha+1 \equiv 2 \pmod 4$, p prime.

Let $a_1, \ldots, a_{n-1}$ be the elements of any Galois field $GF(p^\alpha)$. Define $\chi(0)=0$ and $\chi(a)=1$ or -1 according as $a \neq 0$ is or is not a square in $GF(p^\alpha)$. Then the matrix with elements

$$c_{ij}=\chi(a_i-a_j);\ c_{in}=c_{ni}=1;\ c_{nn}=0;\ i, j=1, \ldots, n-1$$

is a C-matrix since $\sum_{a \in GF(p^\alpha)} \chi(a)\chi(a+b)=-1$ for $b \neq 0$.

However, C-matrices do not exist for all $n \equiv 2 \pmod 4$, as a consequence of the following theorem.

5.2 Theorem. A necessary condition for the existence of a square rational matrix Q of order $q \equiv 2 \pmod 4$ satisfying $Q'Q=mI_q$, m integer, is that m is a sum of two squares of integers.

Proof. By Lagrange's four-square theorem we may write $m=m_1^2+$ $+m_2^2+m_3^2+m_4^2$ with m_1, m_2, m_3, m_4 integers. Put

$$M=\begin{pmatrix} m_1 & -m_2 & -m_3 & -m_4 \\ m_2 & m_1 & -m_4 & m_3 \\ m_3 & m_4 & m_1 & -m_2 \\ m_4 & -m_3 & m_2 & m_1 \end{pmatrix} \text{ and } Q=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with square A of order 4. It follows that $M'M = mI_4$. Any row of M may be multiplied by -1 without altering this property. Since det $M \neq 0$, it is not possible that det $(A-M)=0$ for all possible choices of M. Hence we may assume det $(A-M) \neq 0$. We now prove for $Q^* = D - C(A-M)^{-1}B$ that $Q^{*\prime}Q^* = mI_{q-4}$ by calculating in two ways the matrix product

$$\begin{pmatrix} -B'(A'-M')^{-1} & I_{q-4} \end{pmatrix} \begin{pmatrix} A' & C' \\ B' & D' \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} -(A-M)^{-1}B \\ I_{q-4} \end{pmatrix}.$$

Indeed, denote the factors by X, Y, Z, U, then

$$X(YZ)U = B'(A'-M')^{-1}m(A-M)^{-1}B + mI_{q-4},$$
$$(XY)(ZU) = B'(A'-M')^{-1}M'M(A-M)^{-1}B + Q^{*\prime}Q^*.$$

Thus we come to a matrix Q^* of order $q-4$, satisfying the conditions of the theorem. By iteration a matrix of order 2 is obtained, hence m is a sum of two squares of rationals. Since m is an integer we conclude that m is a sum of two squares of integers.

5.3 **Remark.** Application of the theorem to

$$Q = \begin{pmatrix} 0 & e' \\ e & R \end{pmatrix}, \ R'R + J = (n-1)I, \ R \text{ rational},$$

makes the resemblance to the Bruck–Ryser theorem [6] on $R'R - J = (n-1)I$ apparent.

5.4 **Theorem.** For any C-matrix of the form

$$C = \begin{pmatrix} 0 & e' \\ e & D \end{pmatrix}, \text{ the matrix } P = \begin{pmatrix} e & \frac{1}{2}(J-D-I) \\ 0 & \frac{1}{2}(J-D+I) \end{pmatrix}$$

is the (0, 1) incidence matrix of a balanced incomplete block design $(v, b, k, r, \lambda) = (n, 2n-2, \frac{1}{2}n, n-1, \frac{1}{2}n-1)$.

Proof. $C^2 = (n-1)I$ implies $De = 0$ and $D^2 + J = (n-1)I$. We check the basic properties of the incidence matrix of (v, b, k, r, λ) configurations:

$$P'P = \begin{pmatrix} n-1 & (\frac{1}{2}n-1)e' \\ (\frac{1}{2}n-1)e & \frac{1}{2}nI + (\frac{1}{2}n-1)J \end{pmatrix}, \ (1 \ e')P' = \frac{1}{2}n(e' \ e').$$

5.5 **Remark.** In view of the existence of a Hadamard matrix of order 92, it would be interesting to know whether Paley's construction may be reversed in order to obtain a C-matrix of order 46, the smallest order which is not covered by 5.1 and 5.2 and for which the existence of C-matrices is as yet undecided.

6. *Results on $n(r)$*

6.1 **Lemma.** Let B be a B-matrix of order n whose smallest eigenvalue λ_0 has multiplicity $n-r$ and whose other eigenvalues are $\lambda_1, \ldots, \lambda_r$.

Then

$$\lambda_0 \geqslant -\sqrt{\frac{(n-1)r}{n-r}},$$

equality holding if and only if $\lambda_1 = \ldots = \lambda_r$.

Proof. The consistency of the equations

$$\lambda_1 + \ldots + \lambda_r = \operatorname{tr} B - (n-r)\lambda_0 = -(n-r)\lambda_0$$
$$\lambda_1^2 + \ldots + \lambda_r^2 = \operatorname{tr} B^2 - (n-r)\lambda_0^2 = n(n-1) - (n-r)\lambda_0^2$$

yields $(n-r)^2\lambda_0^2 \leqslant rn(n-1) - r(n-r)\lambda_0^2$, from which the assertion follows.

6.2 Lower bound. $n(r) \geqslant 2r-2$.

Proof. Consider the ladder graph, i.e. the $(2r-2)$-graph consisting of $r-1$ pairs of connected points. The corresponding B-matrix is

$$\begin{pmatrix} J-I & J-2I \\ J-2I & J-I \end{pmatrix}.$$

By elementary methods the eigenvalues are found to be -3 (with multiplicity $r-2$), 1 (with multiplicity $r-1$), and $2r-5$. Therefore $A = I + \frac{1}{3}B$ is the Gram matrix of $2r-2$ vectors in R_r which form an equilateral $(2r-2)$-tuple in E_{r-1} with distance arccos $\frac{1}{3}$. This proves the assertion.

The C-matrices considered in section 5 provide examples of values of r for which better bounds are available. As a consequence of 5.1 we have:

6.3 Lower bound. If $2r-1 \equiv 1 \pmod 4$ is a prime power then $n(r) \geqslant 2r$.

We shall now prove that for $r=5$ this result is best possible.

6.4 Theorem. $n(5) = 10$.

Proof. Suppose $n(5) > 10$. Then there exists a B-matrix of order 11, B_{11} say, with smallest eigenvalue λ_0 of multiplicity 6. From 6.1 it follows that $\lambda_0 \geqslant -5/\sqrt{3}$. On the other hand $I_{11} - B_{11}/\lambda_0$ is the Gram matrix of 11 vectors in R_5. Therefore, B_{11} has a principal submatrix B_7 of order 7 with smallest eigenvalue λ_0 of multiplicity 2. Now table 4.2 contains only type 26 to meet these conditions. However it is easily checked by inspection that type 26 is not extendable to a matrix of order 8 with threefold smallest eigenvalue λ_0. Thus we have $n(5) \leqslant 10$ and by 6.3 the theorem is proved.

For the case $r=6$ we shall need B-matrices of order 8 with smallest eigenvalue of multiplicity $\geqslant 2$. A complete table for $n=8$ is superfluous for our purpose since for the proof of the following theorem 6.5 the only information needed is that such B-matrices do not exist for smallest eigenvalue $\lambda_0 > -3$, $\lambda_0 \neq -1$. This is easily checked by inspection of the relevant types of table 4.2.

6.5 Theorem. $n(6)=16$.

Proof. Example 2.1 provides a B-matrix B_{16} of order 16 with smallest eigenvalue -3 of multiplicity 10. Now suppose $n(6)>16$, then a B-matrix B_{17} of order 17 exists with smallest eigenvalue λ_0 of multiplicity 11. From 6.1 it follows that $\lambda_0 \geqslant -\frac{4}{11}\sqrt{66}$. On the other hand B_{17} must have a principal submatrix B_8 of order 8 with this same smallest eigenvalue λ_0 of multiplicity 2. As remarked above there is no such B_8. This proves the theorem.

In the case $r=5$ the largest equilateral point set is furnished by a C-matrix. The following lower bound implies, as did already theorem 6.5, that this is not the case for $r=7$.

6.6 Lower bound. $n(7)\geqslant 28$.

Proof. We construct 28 vectors, spanning R_7, of length $\sqrt{3}$, such that the inner product of any pair equals 1 or -1. For that purpose we first remark that the vectors

$$(1, 1, 1, 0, 0, 0, 0),\ (1, -1, -1, 0, 0, 0, 0),\ (-1, 1, -1, 0, 0, 0, 0),\ (-1, -1, 1, 0, 0, 0, 0)$$

have that property. Secondly we observe that the same holds for the row vectors of the 7×7 incidence matrix of the finite projective geometry of order 2, since any row contains 3 ones and 4 zeros and any pair of rows has 1 one in common. Combining these observations we obtain the desired 28 vectors in R_7. In fact they are the projections on the hyperplane $x_1=1$ of the 28 half sums of the pairs of row vectors of the normalized Hadamard matrix of order 8.

6.7 Remark. By 6.1 the existence of this 28-tuple implies that the corresponding B-matrix has, apart from a 21-fold eigenvalue -3, a 7-fold eigenvalue 9. We conjecture that $n(7)=28$. Indeed, the existence of a B-matrix of order 29 with 22-fold smallest eigenvalue $\lambda_0 \geqslant -\sqrt{\frac{196}{22}} > -3$ seems implausible.

6.8 Remark. Trivially 6.6 implies $n(8)\geqslant 28$. It is not possible to extend any further the set of 28 vectors, mentioned in 6.6, in order to obtain a better bound for $n(8)$. Indeed, place this set in the hyperplane $x_8=0$ of R_8, then it is easily seen that no vector of R_8 has inner product a or $-a$, $a\neq 0$, a constant, with each of the 28 vectors.

6.9 Remark. From 6.1 it follows that for $r\geqslant 9$ there do not exist B-matrices whose smallest eigenvalue with multiplicity $n-r$ equals -3, and whose other eigenvalues are all equal.

6.10 Remark. It is not difficult to construct examples for other values of r showing that $n(r)>2r$. For instance the blockmatrix of order

64 consisting of the blocks B_{16} (of example 2.1) on the diagonal and $I_{16}-B_{16}$ elsewhere has two eigenvalues -7 and 9 with multiplicities 36 and 28 respectively. This implies $n(28)\geqslant 64$.

7. *Equilateral point sets in E_{r-1}*

From sections 4 and 6 it follows that E_4 contains the following equilateral point sets of order $\geqslant 7$:

(i) a 7-tuple with distance 67°25′7″ whose graph is type 26 of 4.2,
(ii) a 7-tuple with distance 73°22′8″ whose graph is the complement of type 26 of 4.2,
(iii) an 8-tuple with distance arccos $\frac{1}{3}$ whose graph is the ladder 8-graph,
(iv) a 10-tuple with distance arccos $\frac{1}{3}$ whose graph is the Petersen graph.

Furthermore we found the following equilateral point sets:

(v) a 16-tuple in E_5 with distance arccos $\frac{1}{5}$ and matrix 2.1,
(vi) a 28-tuple in E_6 with distance arccos $\frac{1}{3}$.

We add some geometrical comments.

7.1 Lemma. The existence of n points in S_{r-1}, spanning that sphere and having only two distinct spherical distances α and β, implies the existence in E_r of an equilateral n-tuple provided $\cos\alpha+\cos\beta<0$, and of an equilateral $(n+1)$-tuple provided

$$\cos\alpha+\cos\beta=2\cos\alpha\cos\beta<0.$$

Proof. By $\cos\alpha+\cos\beta<0$, we may determine R and φ such that

$$1-\cos\alpha=R^2(1-\cos\varphi),\ 1-\cos\beta=R^2(1+\cos\varphi),\ R>1,\ 0<\varphi<\pi.$$

Now imbed the unit sphere S_{r-1} with its n points in the r-dimensional sphere S_r of radius R, at the euclidean distance $\sqrt{R^2-1}$ from the origin. Then on S_r the n points have spherical distances φ and $\pi-\varphi$. These distances equal the spherical distance of the small sphere to its poles on S_r if and only if $R\sin\varphi=1$, which is equivalent to

$$\cos\alpha+\cos\beta=2\cos\alpha\cos\beta.$$

By transition from spherical to elliptic geometry the lemma is proved.

7.2 Solution. The equilateral 28-tuple in E_6.

The end points of the unit vectors in R_8 are the 8 vertices of a regular simplex of 7 dimensions α_7, contained in the hyperplane $\sum_{i=1}^{8} x_i=1$. We shift this hyperplane over the euclidean distance $1/\sqrt{8}$ in order to situate α_7 in the hyperplane $\sum_{i=0}^{8} x_i=0$. Then the 28 midpoints $P_{h,i}$, $(h<i;\ h, i=1, \ldots, 8)$ of the edges of α_7 have coordinates $P_{1,2}=(3, 3, -1, -1, -1, -1, -1, -1)$, etc., up to a common factor. Connect the origin with all points $P_{h,i}$, then each pair of lines has angle arccos $\frac{1}{3}$. Therefore, the lines

constitute an equilateral 28-tuple in E_6. Its graph is the complement of the graph obtained by putting the 28 points $P_{h,i}$ in the upper left corner of a chessboard and by connecting $P_{h,i}$ and $P_{j,k}$ whenever their index pairs have one number in common.

7.3 Remark. The 28-tuples of 7.2 and 6.6 are the same. The relation to Gosset's semi-regular polytopes (cf. Coxeter [7], p. 202) is as follows. Our equilateral 28-tuple in E_6 consists of the lines connecting the 28 pairs of opposite vertices of the 7-dimensional 3_{21}. Also it is obtained by the process of Lemma 7.1 from the 27 points of the 6-dimensional 2_{21}, since these form a spherical two-distance set with $\cos\alpha=\frac{1}{4}$ and $\cos\beta=-\frac{1}{2}$.

7.4 Solution. The equilateral 16-tuple in E_5.

The subset of the 28 points $P_{h,i}$ of 7.2 that satisfies, apart from $\sum_{i=0}^{8} x_i=0$, the condition $x_1=x_2$ contains the 16 points $P_{1,2}$ and $P_{j,k}$, $(j<k; j, k=3, \ldots, 8)$. The lines connecting the origin with these points span an R_6 and hence constitute an equilateral 16-tuple in E_5 with distance arccos $\frac{1}{3}$. Its graph is the corresponding subgraph of the graph mentioned in 7.2.

A direct equivalent representation of this 16-tuple is the following. Consider the measure polytope (hypercube) γ_6 in R_6 whose 64 vertices have coordinates $(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1)$. Select the 32 vertices of the half measure polytope $h\gamma_6$ by allowing only even numbers of negative signs. Connecting the opposite vertices we again obtain an equilateral 16-tuple in E_5. If we represent it by $(1, 1, 1, 1, 1, 1)$ and the 15 permutations of $(1, 1, 1, 1, -1, -1)$ we obtain the graph referred to above. An equivalent graph is obtained when the 16-tuple is represented and renamed $Q_{h,i}$, $(h, i=1, 2, 3, 4)$ according to

$$\begin{aligned}
&(1,1,1,1,1,1),(-1,-1,1,1,1,1),\\
&\qquad(-1,1,-1,1,1,1),(1,-1,-1,1,1,1),\\
&(1,1,1,1,-1,-1),(-1,-1,1,1,-1,-1),\\
&\qquad(-1,1,-1,1,-1,-1),(1,-1,-1,1,-1,-1),\\
&(1,1,1,-1,1,-1),(-1,-1,1,-1,1,-1),\\
&\qquad(-1,1,-1,-1,1,-1),(1,-1,-1,-1,1,-1),\\
&(1,1,1,-1,-1,1),(-1,-1,1,-1,-1,1),\\
&\qquad(-1,1,-1,-1,-1,1),(1,-1,-1,-1,-1,1).
\end{aligned}$$

Then its graph is the complement of the graph obtained by connecting $Q_{h,i}$ and $Q_{j,k}$ if $h=j$ and if $i=k$. This is precisely the representation corresponding to the B-matrix of example 2.1.

7.5 Remark. The 16-tuple is also obtained by applying Lemma 7.1 to the 15 midpoints of the edges of a regular simplex α_5. Indeed, on S_4 these form a two-distance set with $\cos\alpha=-\frac{1}{2}$ and $\cos\beta=\frac{1}{4}$.

7.6 Solution. The equilateral 10-tuple in E_4.

The subset of the 28 points $P_{h,i}$ of 7.2 that satisfies the condition $x_1=x_2=x_3$ contains the 10 points $P_{j,k}$, $(j<k; j, k=4, \ldots, 8)$. They lead to an equilateral 10-tuple in E_4 with distance arccos $\frac{1}{3}$. Their graph, the corresponding subgraph of the graph mentioned in 7.2, is the selfcomplementary Petersen graph.

A direct equivalent representation of this 10-tuple is obtained by applying Lemma 7.1 to the 10 midpoints of the edges of a regular simplex α_4, which on S_3 form a two-distance set with $\cos\alpha=\frac{1}{6}$ and $\cos\beta=-\frac{2}{3}$.

7.7 Remark. Using the representation of 7.4 we observe that the Petersen graph is equivalent to the subgraph consisting of $Q_{4,4}$ and $Q_{j,k}$, $(j, k=1, 2, 3)$. Geometrically this corresponds to the following construction. The sphere S_3 is fibered over a great sphere S_2 as base space with great circles S_1 as fibers. The fibers issuing from a small circle on S_2 form a torus. Take the radius of this small circle equal to $\frac{1}{2}\sqrt{2}$, then the radii of the torus are equal. Therefore it is possible to select on the torus 9 points that constitute a two-distance set on S_3 with distances $\cos\alpha=\frac{1}{4}$ and $\cos\beta=-\frac{1}{2}$. Applying Lemma 7.1 we again obtain the equilateral 10-tuple in E_4.

7.8 Solution. The ladder 8-tuple in E_4.

The ladder graph of the equilateral 8-tuple in E_4 with distance arccos $\frac{1}{3}$ is equivalent to the subgraph formed by Q_{1i} and Q_{2i}, $(i=1, 2, 3, 4)$ of the 16-graph considered in 7.4. This 8-tuple was obtained by BLUMENTHAL and KELLY ([4], p. 104) from the 8-vertices of a regular cross polytope β_4 by the process of Lemma 7.1.

7.9 Remark. The subset of $P_{h,i}$ of 7.2 that satisfies $x_1=x_2=x_3=x_4$ contains the 6-points $P_{j,k}$, $(j<k; j, k=5, 6, 7, 8)$. It furnishes an equilateral 6-tuple in E_3 whose graph is equivalent to the ladder 6-graph. This 6-tuple, which is obtained directly from the midpoints of a regular simplex α_3, occurs in HAANTJES [10].

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Added in proof:

17. FEJES TÓTH, L., Distribution of points in the elliptic plane, Acta Math. Acad. Sci. Hungar. **16**, 437–440 (1965).
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Professor Coxeter kindly informed the authors that the coordinates for 3_{21}, found in sections 7.2 and 6.6, already appeared in COXETER [18] and [19] respectivily.