Optim Lett (2012) 6:211–217
DOI 10.1007/s11590-010-0264-0

ORIGINAL PAPER

New weighing matrices constructed from two circulant submatrices

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Received: 13 May 2010 / Accepted: 10 November 2010 / Published online: 26 November 2010

Abstract A number of new weighing matrices constructed from two circulants and via a direct sum construction are presented, thus resolving several open cases for weighing matrices as these are listed in the second edition of the Handbook of Combinatorial Designs.

Keywords Weighing matrices · Algorithm · String sorting

1 Introduction

A square $n \times n$ matrix with elements from $\{-1, 0, 1\}$ such that $WW^t = kI_n$ is called a weighing matrix of order n and weight k, denoted by $W(n, k)$. The papers [11] and [12] contain a number of conjectures and constructions for weighing matrices.

In this paper we focus our attention on weighing matrices constructed from two circulants. The two circulants construction for weighing matrices is described in the theorem below, taken from [4].

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Theorem 1 *If there exist two circulant matrices A, B of order n, with $0, \pm 1$ elements, satisfying $AA^t + BB^t = f I_n$ and f is an integer, then there exists a $W(2n, f)$, given as*

$$W(2n, f) = \begin{pmatrix} A & B \\ -B^t & A^t \end{pmatrix} \text{ or } W(2n, f) = \begin{pmatrix} A & BR \\ -BR & A \end{pmatrix}$$

where R is the square matrix of order n with $r_{ij} = 1$ if $i + j - 1 = n$ and 0 otherwise.

See [9] for a description of the problem of searching for weighing matrices as an Optimization problem.

1.1 Applications of weighing matrices

Weighing matrices and optimal weighing designs have long been studied for their use in statistical experiments as first studied by Hotelling [5] and later by Raghavarao [13] and others [11]. These weighing matrices may then used for Coding Theory purposes, i.e. to construct linear codes with desirable properties, see for example [1]. Moreover, the importance of weighing matrices in other areas has been exhibited in the field of quantum information processing [3]. A connection between real world applications and weighing matrices was given in [11], where weighing matrices were used in multiplexing optics.

2 Previous work

The state-of-the art concerning weighing matrices constructed from two circulants until 1999, is described in Lemma 11 of [12]. It is well-known that if the Diophantine equation $a^2 + b^2 = k$ has no solutions, then there do not exist $W(2n, k)$ constructed from two circulants, and therefore we focus our attention to the permissible odd values of n, i.e. values of n such that the Diophantine equation $a^2 + b^2 = k$ has solutions. In a recent series of papers [6–8], the authors investigate weighing matrices constructed from two circulants for the following four classes of large weights: $W(2n, 2n - 5)$, $W(2n, 2n - 9)$, $W(2n, 2n - 13)$, $W(2n, 2n - 17)$. Using a new algorithm based on string sorting, the authors construct a number of new weighing matrices constructed from two circulants, that were listed as open cases in Lemma 11 of [12]. In addition, Lemma 11 of [12] contains a significant number of other open cases which do not fall in any of the four classes of large weights mentioned above.

One way to reduce the computational complexity of the search for $W(2n, 2n - \alpha)$ weighing matrices constructed from two circulants is to exploit structural patterns for the location of the α zeros in the two arrays $[a_1, \ldots, a_n]$ and $[b_1, \ldots, b_n]$. In this context, a $(p, \alpha - p)$ structural pattern is a statement of the form:

there are p zeros in $[a_1, \ldots, a_n]$ and $\alpha - p$ zeros in $[b_1, \ldots, b_n]$.

In this paper we construct a number of new weighing matrices constructed from two circulants with a $(p, \alpha - p)$ structural pattern.

Using the new sequences of Sect. 3, we prove the following main theorem of this paper.

Theorem 2 *There exist weighing matrices $W(2n, k)$ constructed from two circulant submatrices for the following* 16 *pairs of values of n, k:*

k (weight)	*n (order)*
37	*29*
41	*33, 35*
45	*35*
49	*35*
50	*27, 29*
53	*37*
58	*31, 33, 35, 37*
61	*37*
68	*35, 37*
73	*37*

Remark 1 All the 16 cases of Theorem 2 were listed as open in Lemma 11 of [12].

Remark 2 The following 10 new weighing matrices in Theorem 2, were listed as unresolved in the second edition of the Handbook of Combinatorial Designs [2], in the chapter titled *Orthogonal Designs*, (Table 2.85 square weights $w = 49$, Table 2.86 non-square weights $w = 37, 41, 50, 58$) by R. Craigen and H. Kharaghani:

$$W(58, 37),\ W(66, 41),\ W(70, 41),\ W(70, 49),\ W(54, 50),$$
$$W(58, 50),\ W(62, 58),\ W(66, 58),\ W(70, 58),\ W(74, 58).$$

Remark 3 The following new weighing matrices that we find in [6–8], are listed as unresolved in the second edition of the Handbook of Combinatorial Designs [2], in the chapter titled *Orthogonal Designs*, (Table 2.85 square weights $w = 49, 81$, Table 2.86 non-square weights $w = 37, 45$) by R. Craigen and H. Kharaghani:

- $W(2n, 2n-5)$

  $$W(58, 53), W(66, 61), W(70, 65), W(78, 73), W(86, 81), W(90, 85), W(94, 89)$$

- $W(2n, 2n-9)$

  $$W(50, 41), W(58, 49), W(70, 61), W(74, 65), W(82, 73)$$

- $W(2n, 2n-13)$

  $$W(58, 45), W(62, 49)$$

- $W(2n, 2n-17)$

  $$W(54, 37), W(58, 41), W(62, 45), W(66, 49)$$

Note that Table 2.86 in the Handbook, is limited to weights < 60.

3 Algorithm and new results

We used an adaptation of our string sorting algorithm for weighing matrices $W(2n, 2n-13)$, $W(2n, 2n-17)$ constructed from two circulants [8], to search for the new weighing matrices in this paper. The main choice that we had to make was how to distribute the zeros of a weighing matrix $W(2n, k)$ constructed from two circulants, in the first rows of the two circulant matrices A, B. When the number of zeros $2n-k$ was even, we searched for a $W(2n, k)$ with a $\left(\frac{2n-k}{2}, \frac{2n-k}{2}\right)$ structural pattern. When the number of zeros $2n-k$ was odd, we searched for a $W(2n, k)$ with a $\left(\frac{2n-k+1}{2}, \frac{2n-k-1}{2}\right)$ structural pattern. Sometimes we searched for less balanced patterns, such as the case $W(2 \cdot 37, 53)$ for instance.

Example 1 We illustrate with a specific example taken from [9] the construction of $W(2 \cdot 189, 9)$. For the definition of PAF vector, also see [9]. The sparse encoding used in the structural patterns, is defined as the concatenation of the non-zero values of the PAF vector, as well as their positions in the vector, together with the two qualifiers: p for the position, v for the value. To illustrate the sparse encoding of the PAF vector with a large order example, we note that the PAF vector of the weighing matrix $W(2 \cdot 189, 9)$ given in the results section, has sparse encoding equal to

$$p27vm1p36v1p63v1p90v1$$

where the minus sign $-$ has been replaced by the qualifier m, for reasons of algorithmic efficiency. Unraveling this sparse encoding, we obtain the entire PAF vector corresponding to it:

$$[\underbrace{0, \ldots, 0}_{\text{zeros1...26}}, -1, \underbrace{0, \ldots, 0}_{\text{zeros28...35}}, 1, \underbrace{0, \ldots, 0}_{\text{zeros37...62}}, 1, \underbrace{0, \ldots, 0}_{\text{zeros64...89}}, 1, \underbrace{0, \ldots, 0}_{\text{zeros91...94}}]$$

□

For lengths $n \leq 31$ we used serial C programs. For lengths $n \geq 33$ we used parallel C programs. All computations were performed remotely at SHARCnet high-performance computing clusters. The prototype C programs have been generated at the CARGO Lab of Wilfrid Laurier University. The results we obtained are given below using the standard notation where $+$ stands for a $+1$ and $-$ stands for a -1.

```
W(2*27,50)  structural pattern (2,2)

[+ 0 + 0 + + + + - + - + + + - + + + - + - - - - + - -]
[0 + 0 + + - - - + + + + - + + - - + - + + - + + + - -]

W(2*29,37)  structural pattern (11,10)

[- - 0 - + 0 + 0 - 0 + + + 0 0 - - - 0 - 0 0 + - 0 - 0 - -]
[+ + - 0 - + 0 0 + - 0 0 + - - + 0 0 + - - + 0 - + 0 0 - -]

W(2*29,50)  structural pattern (4,4)

[0 - - + - + - + 0 + + + 0 - + + - 0 + + + + + + - - + - -]
[- + + + + - - - + + 0 + - + + - - + 0 + - + 0 + + - + 0 -]
```

```
W(2*31,58)  structural pattern (2,2)

[+ + + + - + - + + - - + - - + - - - 0 - - - 0 + + - + - + - -]
[+ + - + - + + + + + 0 - - - 0 + + - + + + - + + - + + + - - -]

W(2*33,41)  structural pattern (13,12)

[0 - 0 0 - + + - 0 0 + - 0 - + + + - 0 0 - + 0 + 0 0 0 0 - - - - -]
[+ 0 + - 0 0 0 0 0 0 - - + 0 - - + 0 - + - 0 + - 0 0 + + - - - - -]

W(2*33,58)  structural pattern (4,4)

[- + + - - + 0 - + - + - + 0 0 - + + - - - - + - 0 + - + - - - - -]
[0 + + - + - + + + - 0 0 - + + - + + 0 + + + - - + + + - - - - - -]

W(2*35,41)  structural pattern (15,14)

[+ + 0 0 0 0 + 0 0 - + + 0 0 - + + - + 0 0 + + 0 0 0 0 + + 0 - - - - -]
[0 0 0 0 - + - + - + 0 - + 0 - + + - 0 0 0 - 0 0 + - 0 0 + 0 - - - - -]

W(2*35,45)  structural pattern (13,12)

[+ 0 0 0 - + 0 0 + - - + + + - 0 + - 0 0 - - + 0 - 0 0 0 0 - - - - - -]
[+ + + - 0 0 0 0 0 + 0 - + - + - - - + 0 0 + - + 0 0 - + 0 0 - - - - -]

W(2*35,49)  structural pattern (11,10)

[0 + 0 - + + + - 0 + 0 + 0 - + 0 0 + 0 - + - 0 0 + - + 0 + - - - - - -]
[+ + - - - 0 + + 0 - 0 0 - 0 + - + 0 - 0 - + - 0 0 + 0 - + - - - - - -]

W(2*35,58)  structural pattern (6,6)

[0 + 0 + - - + - - + + 0 - - + + - 0 + 0 - + - 0 - + - + - - - - - - -]
[+ + + - 0 - 0 + - + - - + - + 0 + - - + + 0 + 0 - + + 0 - - - - - - -]

W(2*35,68)  structural pattern (1,1)

[+ + - + - + + + - - + + + + - - + - + 0 - + - - - + - + + - - - - - -]
[+ + + + - + + - - 0 - - + - + - + - - + + - - - + - - - + - - - - - -]

W(2*37,53)  structural pattern (8,13)

[+ + - + - - - + + 0 0 - 0 - + + - - 0 + 0 + - 0 0 + - - 0 + - - - - - - -]
[+ + 0 0 + 0 - + - 0 0 + - + 0 - + 0 0 + - + - 0 + 0 0 0 0 + - - - - - - -]

W(2*37,58)  structural pattern (8,8)

[0 + - + - 0 - + + + - 0 0 - + 0 0 + + - - + + 0 - - + 0 + + - - - - - - -]
[0 + - + 0 - - + - + - 0 0 - + 0 0 - - + - + 0 - + + - 0 + + - - - - - - -]

W(2*37,61)  structural pattern (6,7)

[0 + - - + - + + - + - + 0 + - + 0 + - - + + - - - 0 0 + + 0 - - - - - - -]
[+ 0 - + + - - + - 0 + 0 - 0 0 + - - 0 - + - + + - 0 - + + + - - - - - - -]

W(2*37,68)  structural pattern (3,3)

[+ + - + - 0 + 0 - - 0 + + + + - + - + + + - - + - - + - + + - - - - - - -]
[+ - + - + 0 + 0 - - 0 + - + - - - - + + - - + + - - + + - + - - - - - - -]
```

```
W(2*37,73)  structural pattern (1,0)

[+ + - - - + - + - + + - + - - 0 - - + - + + - + - + - - - + + - - - - - -]
[- + - + + + - + + - - + + + + - - + + + + - - + + - + + + - + - - - - - -]
```

4 Additional new results for weight 49 via direct sums

In this section we show that using the new results of the previous section and/or other recent results, we can obtain additional new weighing matrices via direct sums. Let A_1 be an $n_1 \times m_1$ matrix and A_2 be an $n_2 \times m_2$ matrix. The direct sum of A_1 and A_2 is an $(n_1 + n_2) \times (m_1 + m_2)$ matrix defined by

$$A \oplus B = \begin{pmatrix} A & 0_{n_1,m_2} \\ 0_{n2,m_1} & B \end{pmatrix}$$

where $0_{n_1,m_2}$ and $0_{n2,m_1}$ are matrices of zeros.

For weighing matrices $W(n_1, w)$, $W(n_2, w)$ of the same weight, it is easy to see that $W(n_1, w) \oplus W(n_2, w) = W(n_1 + n_2, w)$. We illustrate with four examples that yield four new weighing matrices of weight 49:

1. $W(57, 49) \oplus W(58, 49)$ gives a $W(115, 49)$, which was listed as open in [2];
2. $W(57, 49) \oplus W(62, 49)$ gives a $W(119, 49)$, which was listed as open in [2];
3. $W(57, 49) \oplus W(66, 49)$ gives a $W(123, 49)$, which was listed as open in [2];
4. $W(57, 49) \oplus W(70, 49)$ gives a $W(127, 49)$, which was listed as open in [2].

Note that a circulant weighing matrix $W(57, 49)$ exists for $q = 7$ via the well-known theorem, see [12], regarding circulant weighing matrices of the form $W(q^2+q+1, q^2)$, for q a prime power.

5 Conclusion

In this paper, we have constructed 16 new weighing matrices constructed from two circulants with a $(p, \alpha - p)$ structural pattern, which were listed as open in Lemma 11 of [12] via an adaptation of our string sorting algorithm for weighing matrices [8]. The problem of searching for weighing matrices constructed from two circulant submatrices can be also phrased as a Combinatorial Optimization problem, as shown in [10], thus allowing one to employ Combinatorial Optimization methods to solve it.

Acknowledgments We wish to thank the Editor and the Referees for their constructive comments and suggestions that led to an improved presentation of this paper.

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