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Note

A Construction of D-Optimal Designs for $N \equiv 2 \bmod 4$

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D-optimal design of order 6 is used to construct D-optimal designs of order 42 and 66. © 1987 Academic Press, Inc.

Recently Chadjipantelis *et al.* [1] constructed a number of D-optimal designs by using circulant blocks of size 3 and 5. In this note we will construct D-optimal designs of order 42 and 66 by using (not necessarily circulant) blocks of size 7 and 11.

It was erroneously reported in [1] that design of order 66 was constructed there for the first time. Actually Yang [4] already had constructed a design of order 66.

Since it has not been reported yet, we take this opportunity to announce that D-optimal design of order 82 can be easily constructed from recent design given by Trung [3].

We will use notations used in [1]. Refer to [2] for more information on D-optimal designs.

THEOREM 1. *Let A be any $(-1, 1)$ matrix of order 7 satisfying $AA^{\mathrm{T}} = 8I_7 - J_7$, $AJ_7 = J_7 A = J_7$. Then $R = \left(\begin{smallmatrix} R_1 & R_2 \\ -R_2^{\mathrm{T}} & R_1^{\mathrm{T}} \end{smallmatrix}\right)$ is a D-optimal design of order 42, where $R_1 = A \cdot (J_3 - I_3) + J_7 \cdot I_3$ and $R_2 = A \cdot (J_3 - 2I_3)$.*

Proof. Since J_3, $J_3 - 2I_3$ are circulant, $A^{\mathrm{T}}A = AA^{\mathrm{T}}$ and $J_7 A = AJ_7$ it follows that $R_1 R_2 = R_2 R_1$ and $R_i R_i^{\mathrm{T}} = R_i^{\mathrm{T}} R_i$, $i = 1, 2$. Therefore it is sufficient to show that $R_1 R_1^{\mathrm{T}} + R_2 R_2^{\mathrm{T}} = 40I_{21} + 2J_{21}$. But $R_1 R_1^{\mathrm{T}} + R_2 R_2^{\mathrm{T}} = 7J_7 \cdot I_3 + J_7 \cdot (J_3 - I_3) + J_7 \cdot (J_3 - I_3) + AA^{\mathrm{T}} \cdot (J_3 - I_3)^2 + AA^{\mathrm{T}} \cdot (J_3 - 2I_3)^2 = 40I_7 \cdot I_3 + 2J_7 \cdot J_3 = 40I_{21} + 2J_{21}$. Q.E.D.

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THEOREM 2. *Let A be any $(0, 1, -1)$-matrix of order 11 satisfying $A^{\mathrm{T}} = -A$, $AJ_{11} = J_{11}A = 0$ and $AA^{\mathrm{T}} = 11I_{11} - J_{11}$. Then $R = \begin{pmatrix} R_1 & R_2 \\ -R_2^{\mathrm{T}} & R_1^{\mathrm{T}} \end{pmatrix}$ is a D-optimal design of order 66, where*

$$R_1 = (A + I_{11}) \cdot (J_3 - I_3) + (J_{11} - 2I_{11}) \cdot I_3$$

and

$$R_2 = (A + I_{11}) \cdot (J_3 - I_3) + (-A + I_{11}) \cdot I_3.$$

Proof. Since $R_1 R_2 = R_2 R_1$ and $R_i R_i^{\mathrm{T}} = R_i^{\mathrm{T}} R_i$, it is sufficient to show that $R_1 R_1^{\mathrm{T}} + R_2 R_2^{\mathrm{T}} = 64I_{33} + 2J_{33}$. But,

$$\begin{aligned}
R_1 R_1^{\mathrm{T}} + R_2 R_2^{\mathrm{T}} &= (AA^{\mathrm{T}} + I_{11}) \cdot (J_3 - I_3)^2 + 2(J_{11} - 2I_{11}) \cdot (J_3 - I_3) \\
&\quad + (J_{11} - 2I_{11})^2 \cdot I_3 + (AA^{\mathrm{T}} + I) \cdot (J_3 - I_3)^2 \\
&\quad + (-2AA^{\mathrm{T}} + 2I_{11}) \cdot (J_3 - I_3) + (AA^{\mathrm{T}} + I_{11}) \cdot I_3 \\
&= (AA^{\mathrm{T}} + I_{11}) \cdot 2(J_3 - I_3)^2 + (-24I_{11} + 4J_{11}) \cdot (J_3 - I_3) \\
&\quad + (J_{11} - 2I_{11})^2 \cdot I_3 + (AA^{\mathrm{T}} + I_{11}) \cdot I_3 \\
&= (12I_{11} - J_{11}) \cdot (2J_3 + 3I_3) \\
&\quad + (-24I_{11} + 4J_{11}) \cdot (J_3 - I_3) + (7J_{11} + 4I_{11}) \cdot I_3 \\
&= 64I_{33} + 2J_{33}.
\end{aligned}$$

Q.E.D.

Remarks. (1) In Theorem 1, let A be the circulant matrix with first row $(+ + + - + - -)$, then R is a block circulant D-optimal design of order 42 with block size 7.

(2) In Theorem 2, let A be the (skew-type) circulant matrix with first row $(+ - + - - - + + + - +)$, then R is a block circulant D-optimal design of order 66 with block size 11.

(3) We have demonstrated here that non-circulant blocks may be as useful as circulant blocks for constructing D-optimal designs. One can see that for all designs of block size 3 given in [1], $A = J_3$ and non-circulant

$$B = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

also provides D-optimal designs.

A D-Optimal Design of Order 82

Recently Trung [3] constructed a symmetric block design with parameters (41, 16, 6). It is known that such a design cannot be circulant. Let S be the $(-1,1)$-incidence matrix of this design. Then $SS^{\mathrm{T}} = 40I_{41} + J_{41}$. Hence $\begin{pmatrix} S & S \\ -S & S \end{pmatrix}$ is a non-circulant D-optimal design of order 82.

Acknowledgment

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