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Spectral Characteristics of the Linear Systems over a Bounded Time Interval[1]

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Abstract—Consideration was given to the spectral characteristics of the linear dynamic systems over a bounded time interval. Singular characteristics of standard dynamic blocks, transcendental characteristic equations, and partial spectra of the singular functions were studied. Relationship between the spectra under study and the classical frequency characteristic was demonstrated.

1. INTRODUCTION

The spectral characteristics of the linear operators of dynamic systems are widely used to analyze and design the controllers. The frequency stability criterion, design of the correcting blocks by the logarithmic frequency characteristics, and others are regarded as the most important results. Although great progress was made in this area of research, it continues to enlarge. The most recent examples are the methods of reducing the dynamic system models on the basis of the singular values of the Hankel operator and the methods of designing the H_∞-optimal controllers on the basis of the singular values of the transfer function. The control theory usually considers the spectral characteristics of the linear operators over the nonbounded time interval of process development. Despite the fact that these idealizations simplify the mathematical apparatus, this often happens to the detriment of the descriptive aspect of the application.

The present paper studies the spectral characteristics of the convolution operator over the finite time interval $(0, \mathrm{T})$ giving preference to the frequency approach combined with the flip method [1]. The frequency approach plays a special part in the control theory where the gain frequency characteristic (GFC) and phase frequency characteristic (PFC) of the dynamic plants and the Bode and Nyquist diagrams are used extensively to solve various problems of analysis and design of the control systems. However, all of them correspond to the infinite time interval of applying the action, whereas the physical systems operate in a finite time. A desire naturally arises to introduce the counterparts of the frequency characteristics over a bounded time interval and to clarify their relationship with the classical characteristics. This question can be answered in terms of the spectral characteristics of the convolution operator.

The present paper demonstrates that the singular values of this operator considered over the interval of length T make up a discrete set whose points lie on the classical continuous GFC. The longer the interval T, the denser the lines of the discrete operator spectrum. This regularity is nontrivial and does not imitate in all details—as it could be expected—the relation between the discrete and continuous spectra of signals from the theory of Fourier transform. Although upon passing from the infinite time interval to the finite one the spectrum of the convolution operator becomes discrete, its readings on the frequency axis—in contrast to the theory of signals—do not

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become regular. As regards the spectral characteristics of the convolution operator, the following three main problems are studied.

Problem 1. Determination of the characteristic equation (or assembly of equations) for the singular values of the convolution operator. In the matrix calculus, the corresponding finite dimensional counterpart involves a polynomial; here, it will be a transcendental function with an infinite (countable) number of zeros.

Problem 2. Determination of the analytical relationships between the singular values and singular functions. The matrix theory knows only few examples where the components of eigenvectors are expressed explicitly through the eigenvalues. It is known, in particular, that for the Frobenius matrix the eigenvectors make up the Vandermonde matrix of eigenvalues. It is all the more of interest to establish such relations for the infinite dimensional problems.

Problem 3. Determination of the characteristics of the singular functions of the convolution operator and establishment of their relation with the classical characteristics. Continuity of the singular functions does not rule out the possibility of describing them by a finite set of coefficients. For example, the weight function of the scalar system of order n is fully characterized by $2n$ parameters that can be related with the coefficients of the transfer function. The singular function can be regarded as a model of the dynamic system disclosing its throughput over the finite time interval. Like any other time function, it has the amplitude Fourier spectrum. The frequencies of the singular function, that is, its partial spectrum, are related with the topological features of the GFC. We note that since Nyquist's analysis of system stability relies on the GFC topology, the approach developed is in line with the traditions of the automatic control theory.

The study, therefore, is decomposed into three stages such as (i) the study of the quantitative regularities distinguishing the spectral characteristics of the linear dynamic systems over the bounded time interval, (ii) the study of the partial frequencies of the harmonic components of the singular functions and their localization (similar to the Gershgorin circles for the partial spectrum of singular functions, rather than for the operator spectrum), and (iii) the development of numerical algorithms and grapho-analytical methods of determination of the spectral characteristics.

The paper consists of six sections. Section 2 introduces the necessary mathematical models and defines the singular functions of the convolution operator over the finite time interval. Section 3 establishes the important property of symmetricity of the singular functions and other qualitative characteristics simplifying determination of their analytical description. Section 4 is devoted to determining the singular functions on the basis of the frequency approach, the flip operator [1, 2] that plays here an important part.

To underline the topicality and novelty of the study, we note that the existing scientific and educational literature on the control theory says little about these issues. The time characteristics such as the pulse and transient functions of the elementary dynamic blocks are known well, whereas the eigenfunctions and singular functions of the linear operators of the same plants remain a mystery. This matter is studied in Section 5 where the spectra of the standard blocks are determined and the profile of variations of the spectral characteristics of the convolution operator over the interval T is established. This approach is useful in procedural terms because it enables one to predict complex behavior of the dynamic systems where the spectral composition and "specific weight" of the partial components involved in the singular functions vary with decreasing time interval T and on a small time interval become similar to the elementary blocks.

2. LINEAR OPERATORS OF THE DYNAMIC SYSTEM AND THEIR SINGULAR FUNCTIONS

We consider a linear stationary one-input one-output dynamic system obeying the following equations in the state space:

$$\dot{x} = \mathrm{A}x + \mathrm{B}u; \quad y = \mathrm{C}x; \quad x(0) = 0, \tag{2.1}$$

where $x \in R^n$ is the state vector; $\mathrm{A}, \mathrm{B}, \mathrm{C}$ are the constant matrices; and $u(t)$ and $y(t)$ are the respective scalar input and output, $0 \le t \le \mathrm{T}$.

The integral relation

$$y(t) = Su(t) = \int_0^t q(t-\tau)u(\tau)d\tau, \quad 0 \le t \le \mathrm{T}, \tag{2.2}$$

where $q(t) = \mathrm{C}e^{\mathrm{A}t}\mathrm{B}$ is the pulse weight function of the dynamic system, defines the convolution operator S which characterizes the map of the set of inputs acting on the system over the time interval $(0, \mathrm{T})$ onto the set of outputs considered over the same interval. It corresponds to the real-time operation, which is characteristic the majority of problems of the automatic control theory and theory of electric circuits.

The conjugate operator S^* is introduced in the standard manner by $(\nu, Su) = (S^*\nu, u)$, where parentheses stand for the scalar product of the corresponding functions considered over the interval $(0, \mathrm{T})$. The operator S^* is the convolution operator of the conjugate system of differential equations

$$-\dot{\xi}(t) = \mathrm{A}^{\mathrm{T}}\xi(t) + \mathrm{C}^{\mathrm{T}}\nu(t); \quad \widetilde{y}(t) = \mathrm{B}^{\mathrm{T}}\xi(t); \quad \xi(\mathrm{T}) = 0. \tag{2.3}$$

Here, $\nu(t)$ and $\widetilde{y}(t)$ are the scalar input and output, respectively; the initial condition $\xi(0)$ depends on the form of the input $\nu(t)$ and follows the formula $\xi(0) = -\int_0^{\mathrm{T}} e^{\mathrm{A}^{\mathrm{T}}t}\mathrm{C}^{\mathrm{T}} \times \nu(t)dt$. In the integral form, the operator S^* is represented as follows:

$$\widetilde{y}(t) = S^*\nu(t) = \widetilde{y}_0 + \int_0^t q^*(t-\tau)\nu(\tau)d\tau, \quad 0 \le t \le \mathrm{T}, \tag{2.4}$$

where $q^*(t) = \mathrm{B}^{\mathrm{T}}e^{\mathrm{A}^{\mathrm{T}}(\mathrm{T}-t)}\mathrm{C}^{\mathrm{T}}$ is the pulse weight function of the conjugate system. The initial value $\widetilde{y}_0$ is calculated from the condition $y(\mathrm{T}) = 0$ and is $\widetilde{y}_0 = -\int_0^{\mathrm{T}} q(t)\nu(t)dt$. The equality $q^*(t) = q(\mathrm{T}-t)$ is valid for the scalar systems.

In the inverse time $\tau = \mathrm{T} - t$, the equations of the conjugate system coincide with those of the dual system:

$$\dot{\xi}(\tau) = \mathrm{A}^{\mathrm{T}}\xi(\tau) + \mathrm{C}^{\mathrm{T}}\nu(\tau); \quad \widetilde{y}(\tau) = \mathrm{B}^{\mathrm{T}}\xi(\tau); \quad \xi(0) = 0.$$

The one-input one-output linear stationary dynamic systems are self-dual in the sense that their input-output characteristics—the weight and transfer functions—coincide with the corresponding characteristics of the dual systems.

For the zero initial conditions, the rational transfer functions

$$\mathrm{Q}(p) = \mathrm{C}(p\mathrm{E} - \mathrm{A})^{-1}\mathrm{B} = \frac{b(p)}{a(p)} \quad \text{and}$$
$$\mathrm{Q}^*(p) = \mathrm{C}(-p\mathrm{E} - \mathrm{A})^{-1}\mathrm{B} = \mathrm{Q}(-p) = \frac{b(-p)}{a(-p)}, \tag{2.5}$$

which are the Laplace transforms of the corresponding pulse weight functions, correspond to the respective descriptions (2.1) and (2.3) of the original and conjugate systems. One must discriminate

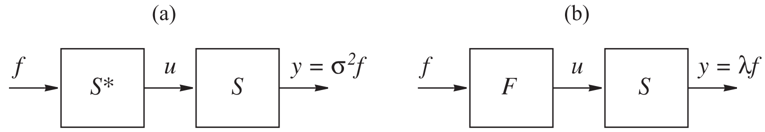

Fig. 1. Variants of defining the singular function of the convolution operator: (a) by the operator of the conjugate system; (b) by the flip operator.

here between the conjugate transfer function $\mathrm{Q}\,(p^*)$ of the system S and the transfer function $\mathrm{Q}^*(p)$ of the conjugate system S^* which coincide only in the special case of harmonic inputs where $p = j\omega$.

Besides the above operators, this paper makes use of the operator

$$F\,f(t) = f(\mathrm{T} - t) \tag{2.6}$$

of mirror time inversion of the function $f(t)$ defined over the interval $(0, \mathrm{T})$. According to the terminology of [1], it will be called the flip operator. Its spectral properties and importance for the theory of dynamic systems were covered in [2]. The flip operator has some special features: it is symmetrical, $F = F^*$, orthogonal, $F^* = F^{-1}$, and involutive, $F^2 = \mathrm{I}$, where I is the identity operator. We also note the nontrivial representation of the conjugate operator as $S^* = FSF^*$, which means that in the case of stationary systems the operator pair S^*, S is related by the homothety with all consequences. The relative simplicity of describing the flip operator in the time domain does not mean that in the frequency domain it is equally simple. This issue is covered in more detail in Section 4.

This paper considers the spectral properties of the operators of the dynamic system (2.1), and first of all the singular values and singular functions of the convolution operator (2.2) over the finite time interval. We present the appropriate mathematical definitions.

In the case of matrix operators, the eigenvalues and singular values are determined from the characteristic equations of the form $|S - \lambda\,\mathrm{I}| = 0$, $|S^*S - \sigma^2\,\mathrm{I}| = 0$. Similar equations appear in the discrete description of the dynamics of linear systems by matrices over the finite time interval.

By passing from matrices to continuous dynamic systems, we note that any dynamic system with the zero initial conditions considered over the finite or semiinfinite time interval filters the input and inevitably changes its form. Therefore, its convolution operator has no eigenfunctions. At the same time, any linear operator has singular functions. There are two approaches to determining the singular operator functions. In addition to the operator S of the system itself, the first approach introduces its conjugate operator S^*.

Definition 1. The functions $f_i(t)$ and $g_i(t)$ satisfying the operator equations

$$SS^*f_i(t) = \sigma_i^2 f_i(t), \quad S^*Sg_i(t) = \sigma_i^2 g_i(t) \tag{2.7a}$$

are called, respectively, the left and right singular functions of the convolution operator S. The arithmetic roots σ_i of the proportionality coefficients in these equalities are called the singular values of the operator S.

Figure 1a illustrates structural interpretation of this definition for the left singular functions. According to it, by the left singular function $f(t)$ is meant the nonzero input passing without form distortion through the serially connected systems S^* and S. The scheme for the right singular functions, which is obtained by permuting the blocks S^* and S, is similar.

We note that the right and left singular functions are related by the equalities

$$S^*f_i(t) = \sigma_i g_i(t), \quad Sg_i(t) = \sigma_i f_i(t) \tag{2.7b}$$

which are the symmetrical notation of Eqs. (2.7a). In applications, the main part is played by the functions $f_1(t)$ and $g_1(t)$ corresponding to the maximal singular value σ_1 and sometimes called the Schmidt maximizing pair.

The second approach to determining the singular functions relies on their extremal properties.

Definition 2. For the given operator S, let us consider the following conditional extremum problem

$$J = \|Sf(t)\| \Rightarrow \text{extr}, \quad \|f(t)\| = 1, \tag{2.8}$$

where $\|\cdot\|$ stands to the function norm. The functions $f_i(t)$ corresponding to the stationary points of this extremal problem are called the singular functions of the operator S, and the corresponding values σ_i of the functional J are called the singular values of the operator S.

The function $f_1(t)$ maximizing globally the functional J is called the main singular function to which corresponds the main singular value σ_1 characterizing the maximum—in the sense of the norm used—"gain" realizable by the given operator.

If one makes use of the quadratic norm of the function $\|f\|_2 = (f,f)^{\frac{1}{2}} = \left(\int\limits_0^{\mathrm{T}} f(t)^2 dt\right)^{\frac{1}{2}}$, then both definitions provide the same result. In the case of other—for example, modular $\|f\|_1 = \int\limits_0^{\mathrm{T}} |f(t)|dt$ or Chebyshev $\|f\|_\infty = \max\limits_t(|f(t)|)$—norms, Definitions 1 and 2 are no more equivalent. In what follows, the paper considers the quadratic norms of functions and the operator norms coordinated with them.

To avoid terminological confusion, we mention the four varieties of spectra encountered in the literature and used in this paper: the amplitude signal spectrum, the modal and operator spectra of the dynamic system, and the partial spectrum of the singular function.

The first variety—*the amplitude signal spectrum*—is widely used in the communication theory and other applications. It is obtained by the Fourier transform and characterizes the amplitudes of the harmonics involved in the signal.

The second variety is *the system modal spectrum* by which is meant the totality of system poles defining the modal components of its free motion. Obviously, it coincides with the spectrum of the matrix A of description (2.1).

The third variety is *the operator spectrum of the dynamic system.* In the functional analysis, by the operator spectrum is usually meant the totality of its eigenvalues corresponding to the eigenfunctions. As was already noted, the convolution operator has no eigenfunctions. We study the spectrum of the symmetrical operator S^*S making up a countable set for the finite dimensional systems. Therefore, wherever below we mention the operator spectrum of the dynamic system, we imply namely this spectrum. In more formal terms: by the operator spectrum of the linear system is meant the totality of singular values of its convolution operator (2.2) considered over the time interval $(0, \mathrm{T})$. According to the general properties of the symmetrical operators, these numbers are real and positive, and the singular functions are pairwise orthogonal.

The fourth variety is *the partial spectrum of the singular function.* As will be shown below, each of the singular functions of the convolution operator of the dynamic system is a linear combination of a finite number of harmonic signals, the modal components of a doubled-order dynamic system. The totality of their frequencies will be called the partial spectrum of the singular function.

In addition to the above classical approaches to determining the singular functions, another, third, approach is possible. It is based on the special property of symmetry inherent into the singular functions of the convolution operator of the scalar stationary systems. We consider this approach in the following section.

3. PROPERTIES OF THE SINGULAR FUNCTIONS

To study the properties of the singular functions, the system of differential equations of the order $2n$ obtained by cascading the original and conjugate systems is put down according to Definition 1 (Fig. 1a). This connection is described by Eqs. (2.1) and (2.3) complemented by the connection equation $u(t) = \widetilde{y}(t)$:

$$\begin{cases} \dot{x} = \mathrm{A}x + \mathrm{B}u; & y(t) = \mathrm{C}x(t); \quad u(t) = \widetilde{y}(t) \\ \dot{\xi}(t) = -\mathrm{A}^{\mathrm{T}}\xi(t) - \mathrm{C}^{\mathrm{T}}f(t); & \widetilde{y}(t) = \mathrm{B}^{\mathrm{T}}\xi(t); \quad f(t) = y(t)/\sigma^2. \end{cases} \tag{3.1}$$

The last relation exists if the singular function of the system S plays the part of the input $f(t)$. According to (2.7), the output $y(t) = SS^*\nu(t)$ differs then from the input only in the amplitude $y(t) = \sigma^2 f(t)$.

The state space matrix of system (3.1) is as follows:

$$\mathrm{M} = \begin{pmatrix} \mathrm{A} & \mathrm{BB}^{\mathrm{T}} \\ -\mathrm{C}^{\mathrm{T}}\mathrm{C}/\sigma^2 & -\mathrm{A}^{\mathrm{T}} \end{pmatrix}. \tag{3.2}$$

We note that a similar matrix appears in the optimal control theory upon minimizing the quadratic functional $\int\limits_0^\infty (x^{\mathrm{T}}\mathrm{Q}x = u^{\mathrm{T}}Ru)dt$. These problems in fact are close in nature, but they also have three essential differences. First, when determining the singular functions, one seeks the maximum of the quadratic functional, and when determining the optimal control, its minimum. Second, the matrices differ in the sign of the left lower block (in the optimal control problems, it is positive). Third, the matrix M involves the unknown parameter σ which should be chosen so as to make the solution of system (3.1) satisfy the boundary conditions $x(0) = 0$, $\xi(\mathrm{T}) = 0$.

These differences are no impediment to the use of the standard control-theoretical apparatus in order to determine the singular functions. It is based on constructing the Riccati differential equation and calculating the coefficients of the feedback matrix that provide the optimal form of the system output. The analytical difficulties of this approach are well known. In the optimal control theory where only one solution is sought and the singular value σ is not required, bulky calculations arise already in the case of analyzing the aperiodic block [3, 4], and with increased system orders the analytical expressions become immense. We also note that the variational methods of the optimal control theory are in bad agreement with the frequency methods, which estranges us from the traditional spectra, that is, GFC and PFC.

An alternative approach to determining the singular functions which is based on using symmetry of a special kind peculiar to the convolution operator and its singular functions [2, 5] is devoid of these disadvantages. The essence of this symmetry becomes clear from the following theorem.

Theorem 1 (on mirror symmetry). *The left and right singular functions $f_i(t)$ and $g_i(t)$ of the convolution operator* (2.2) *that correspond to the simple singular value σ_i and are related by* (2.7b) *satisfy the mirror symmetry condition*

$$f_i(t) = \pm g_i(\mathrm{T} - t). \tag{3.3}$$

The results obtained here and below are proved in the Appendix.

Condition (3.3) implies that the left singular function coincides to sign with the copy of the right singular function taken in the reverse time. The theorem suggests another, third definition of the singular functions of the convolution operator of system (2.1).

Definition 3. The functions $f_i(t)$ and $g_i(t)$ satisfying the operator equations

$$Sf_i(\mathrm{T}-t) = \lambda_i f_i(t), \quad Sg_i(t) = \lambda_i g_i(\mathrm{T}-t), \tag{3.4}$$

are called, respectively, the left and right singular functions of the convolution operator S corresponding to the singular value $\sigma_i = |\lambda_i|$.

In contrast to the arithmetic singular value σ_i, the real coefficient λ_i, which can be both positive and negative, will be called the algebraic singular value. The subscript of the singular functions and numbers will be omitted below for simplicity.

Structural interpretation of equalities (3.4) is transparent. In particular, the first equality means that if the left singular function $f(t)$ taken in the reverse time is input into system (2.1), then the output will be $f(t)$ amplified with the gain λ. This fact is illustrated in Fig. 1b where F denotes the flip operator (2.6) providing the mirror copy of the input in reverse time. Therefore, it follows from Definition 3 that by the singular function of the linear dynamic system is meant the input passing—after its inversion in time—to the output without form distortion.

We note that inversion of the input relative to the middle of the finite control interval (flip operation) does not affect the energy relationships of the inputs and outputs, which in some problems allows one to replace the convolution operator by the symmetrical operator H which is introduced by the factorization $S = HF$ [2]. The eigenfunctions of the operator H coincide with the singular functions of the convolution operator S. Since the quadratic norms of the direct and conjugate operators are identical, the eigenvalues of the operator H are equal to the sought singular values to the sign.

Now, we make use of all three definitions of the singular functions to consider their properties without determining them explicitly. The need for doing so is dictated by the fact that the analytical part of the problem is very complicated and that to solve it some qualitative information may prove useful.

As was already noted, each of the singular functions of the linear system over the infinite time interval is a sinusoid, that is, monoharmonic signal. Over the finite time interval, the structure of the singular function becomes appreciably more complicated and includes several harmonics of different frequencies, that is, becomes a polyharmonic signal. The total number of harmonics is n, and the totality of their frequencies makes up a partial spectrum of the singular function. The following theorem describes the composition of this spectrum in more precise terms.

Theorem 2 (on the partial spectrum). *For the minimum-phase system with the transfer function* $\mathrm{Q}(p) = b(p)/a(p)$*, the right and left singular functions of the convolution operator are linear combinations of a finite number of the harmonics* $h_1(t), h_2(t), \ldots, h_n(t)$*.*

The form of the harmonics corresponding to the singular value σ is defined by the roots of characteristic equation

$$\sigma^2 a(p)a(-p) + b(p)b(-p) = 0. \tag{3.5}$$

The theorem is proved in the Appendix.

The singular value σ is involved in the characteristic equation as a parameter. If its value is not known in advance, then Eq. (3.5) does not allow one to establish the partial spectrum, but imposes on it some qualitative constraints.

Corollary. *For the minimum-phase systems with the irreducible transfer function, the partial spectrum cannot include the zeros and poles of the original system, the zeros and poles of the conjugate system, as well as isolated exponential components.*

According to the corollary, the singular functions have no isolated exponential components, which sets them apart from the "free" system motions. The singular system has no modal components, that is, does not excite them. Some analogy can be found in the inputs with frequencies corresponding to the zeros of the transfer function. They excite the plant eigenmotions, but do not reach the output. The singular functions act in the opposite way: they themselves pass without exciting the modal components. This important property enables one to localize and analyze the harmonic singular functions defined over the finite time interval by means of the frequency characteristics (such as GFC and PFC) determined for the infinite-length intervals. This in essence substantiates the possibility of using here the conventional frequency approach.

To comment the corollary, we note that the optimal control theory relates the form of the solution with the peculiarities of the Hamiltonian matrix (3.2). We recall that the matrix M is called the Hamiltonian matrix if the equality $J^{-1}\mathrm{M}^{\mathrm{T}}J = -\mathrm{M}$ with the matrix $J = \begin{pmatrix} 0 & \mathrm{E} \\ -\mathrm{E} & 0 \end{pmatrix}$ is satisfied [6]. The spectrum of the Hamiltonian matrix has a remarkable property that together with each number μ it has, and with the same multiplicity, the number $-\mu$. Stated differently, it has doubly symmetrical complex numbers.

In the problems of optimal control, the matrix M has no purely imaginary eigenvalues, precisely half of its spectrum belongs to the half-plane $\operatorname{Re}\mu < 0$. This defines uniquely the n-dimensional invariant subspace corresponding to the stable eigenvalues and including the optimal trajectory.

The partial spectrum of singular functions touches upon the imaginary axis—the sign of the left lower block of coefficients manifests itself—but remains double-symmetrical. Therefore, the problem under consideration has its own distinctions.

Determination of the frequencies of the partial components of the singular function, that is, of its partial spectrum, is a difficult problem. Its solution can be facilitated by localizing this spectrum with the use of the following property.

We consider the function $R(p) = \mathrm{Q}(p)\mathrm{Q}(-p)$ of complex variable which coincides with the squared ordinary GFC on the imaginary axis and differs from it in the remaining cases.

Property 1 (spectrum localization). The partial spectrum of the singular system function lies on the complex plane constrained by the conditions

$$\operatorname{Im}(R(p)) = 0, \quad \operatorname{Re}(R(p)) = \sigma^2 \geq 0. \tag{3.6}$$

Let us explain the physical sense of these relations. According to the first definition of the singular function and the form of the transfer functions of the direct, S, and conjugate, S^*, systems, the transfer function of their cascade is $\mathrm{Q}(p)\mathrm{Q}(-p)$. The singular function passing through it retains its form and is just multiplied by the positive coefficient σ^2. In terms of the frequency approach, this is expressed by two conditions (3.6).

The theory knows the series connection of the direct and conjugate systems as the amplitude filter which changes the amplitudes of harmonics, but leaves the phases unchanged because the conjugate system plays the role of the phase compensator. Consequently, all harmonics pass through this connection with retention of their form and identical amplification. Additional comments to this property are left for the Appendix, and an example of using it will be considered in what follows.

Property 2 (boundary conditions). The left singular functions of the convolution operator of system (2.1) corresponding to the algebraic singular value λ satisfy the boundary conditions

$$f(0) = 0, \quad \lambda\dot{f}(0) = q(0)f(\mathrm{T}), \quad \lambda\ddot{f}(0) = \dot{q}(0)f(\mathrm{T}) - q(0)\dot{f}(\mathrm{T}), \dots . \tag{3.7}$$

These formulas are regular and representable in the matrix form whose structure will be explained through the example of the four first conditions:

$$\lambda \begin{pmatrix} f(0) \\ \dot f(0) \\ \ddot f(0) \\ \dddot f(0) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ q(0) & 0 & 0 & 0 \\ \dot q(0) & q(0) & 0 & 0 \\ \ddot q(0) & \dot q(0) & q(0) & 0 \end{pmatrix} \begin{pmatrix} f(\mathrm{T}) \\ -\dot f(\mathrm{T}) \\ \ddot f(\mathrm{T}) \\ -\dddot f(\mathrm{T}) \end{pmatrix}$$

or

$$\lambda \begin{pmatrix} f(0) \\ \dot f(0) \\ \ddot f(0) \\ \dddot f(0) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \mathrm{CB} & 0 & 0 & 0 \\ \mathrm{CAB} & \mathrm{CB} & 0 & 0 \\ \mathrm{CA^2B} & \mathrm{CAB} & \mathrm{CB} & 0 \end{pmatrix} \begin{pmatrix} f(\mathrm{T}) \\ -\dot f(\mathrm{T}) \\ \ddot f(\mathrm{T}) \\ -\dddot f(\mathrm{T}) \end{pmatrix},$$

the second variant of notation is convenient when defining the system equations in the state space.

The boundary conditions for the right singular functions have a similar form and are obtained from the above ones by the replacement $f(t) = g(\mathrm{T} - t)$: for example, $g(\mathrm{T}) = 0$, $\lambda \dot g(\mathrm{T}) = q(0)g(0)$, and so on

The quantities $q(0) = \mathrm{CB}$, $\dot q(0) = \mathrm{CAB}$, $\ddot q(0) = \mathrm{CA^2B}, \ldots$ are called the Markov parameters of the dynamic system. With their aid, one can express an interesting topological property formulated for the left singular functions.

Property 3 (interrelationship between the parameters). If the initial values of the singular function are normalized by the condition $\dot f_i(0) = q(0)$ with the use of the Markov parameters, then their finite values "sweep up" the operator spectrum $f_i(\mathrm{T}) = \lambda_i$ (if the first derivative is zero, then we normalize the second one, and so on).

In this way, the interrelationship between the singular functions and the operator spectrum of the system is established by means of the Markov parameters. Importance of this property for analysis of the singular functions will be illustrated by a simple example. The sinusoidal singular functions of the ideal oscillatory block $\ddot y + k^2 y = u$ have the form $f(t) = \sin(\omega_1 t + \theta_1)\sin(\theta_2) - \sin(\omega_2 t + \theta_2)\sin(\theta_1)$. It follows from the initial conditions that the frequencies and phases of the partial components ω_1, ω_2, θ_1, and θ_2 satisfy the relation

$$\omega_1/\tan(\theta_1) = \omega_2/\tan(\theta_2)$$

which is invariant to the number of the singular function.

Hence, the knowledge of any three parameters of the singular functions enables one to determine the fourth parameter. Similar relationship exists for the singular functions with hyperbolic components. By taking these relationships into account, one can avoid laborious calculations. A similar situation occurs when analyzing system stability by the logarithmic frequency characteristics where the PFC is not used at all. Analysis of the corresponding connection played here a remarkable role.

4. DETERMINATION OF THE SINGULAR FUNCTIONS BY THE FREQUENCY APPROACH

The frequency approach became popular because it can represent in a relatively simple form the fairly complicated relationships arising in analysis of the dynamic systems. To apply it in the case at hand, it is necessary to consider the frequency characteristics not only on the imaginary axis, as it is the case in the conventional frequency analysis, but over the entire complex plane. This refers to both units shown in Fig. 1b (the system itself and the flip operator).

Frequency characteristics of the system. Let us consider the generalized system frequency characteristic $\mathrm{A}(p)$ regarded as the transfer coefficient of the partial sinusoidal and hyperbolic components of signals. It is expressed in terms of the previously introduced function $R(p) = \mathrm{Q}(p)\mathrm{Q}(-p)$ as follows:

$$\mathrm{A}(p) = \sqrt{R(p)}. \tag{4.1}$$

This characteristic coincides with the ordinary GFC only on the imaginary axis where $p = j\omega$. For the hyperbolic signals like $\sinh(at)$, one must substitute $p = a$ into (4.1)

Frequency characteristics of the flip operator. This operator is a linear, but nonstationary object. It does not change the amplitudes of harmonics, its GFC being unity, but shifts the signal in time. The value of shift depends on the initial phase, and cases are possible where the signal remains unchanged at all.

Nonstationary plants are usually described by the parametric transfer functions of time or a parameter. In the Goldfarb method, for instance, the parametric transfer function of the linearized nonlinear element depends on the input amplitude. We use the flip method to introduce the nonstationary unit whose transfer function depends, on the contrary, on the input phase θ.

Let the input be $u(t) = \sin(\omega t + \theta)$, then the output of the flip operator is as follows:

$$u(\mathrm{T} - t) = \sin(\omega\mathrm{T} - \omega t + \theta),$$

where T is the length of the time interval.

To estimate the phase shift, we put down the same signal as $u(\mathrm{T} - t) = \sin(\omega t + \pi - \omega\mathrm{T} - 2\theta)$. It is clear that the supplement $\pi - \omega\mathrm{T} - 2\theta$ describes the PFC of the flip operator

$$\psi(\omega, \theta) = \pi - \omega\,\mathrm{T} - 2\theta. \tag{4.2}$$

In the case of aperiodic signals, analysis of the PFC provides similar results, but sign inversion of the harmonic cannot be taken into account by the phase shift on π.

After these remarks, we proceed to determining the singular values and the singular functions on the basis of the frequency approach. Each singular function $f(t)$ depends on $2n$ frequencies and phases of the partial components, the total number of these unknowns being $2n + 1$ with account for the algebraic singular value. This tells on the number of the linkage equations required to determine them. In some cases, the method of variable elimination allows one to pass to one equation of one parameter which is most appropriately can be the singular value λ. This equation will be called the characteristic equation for the operator spectrum of the dynamic system.

The $2n$ first equations are obtained using the third definition of the singular function:

$$SFf = \lambda f. \tag{4.3}$$

This relation can be treated as the equation of operator balance. For the frequency approach, it is divided into equations of the amplitude and phase balances.

We confine ourselves for simplicity to the singular functions with only sinusoidal components. Over the infinite and semiinfinite time intervals, the system has only such singular functions, and upon passing to the finite interval, they are usually dominating. The amplitude balance equation for these functions is as follows:

$$\mathrm{A}(\omega) - |\lambda| = 0, \tag{4.4}$$

where $\mathrm{A}(\omega)$ is the ordinary system GFC.

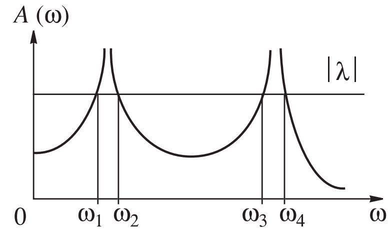

Fig. 2. Graphic interpretation of the amplitude balance equation.

The corresponding equation of phase balance is obtained by taking into account the PFC of the cascaded units in Fig. 1b:

$$\varphi(\omega) + \psi(\omega, \theta) = \arg(\lambda), \tag{4.5}$$

where $\arg(\lambda) = 0$ for $\lambda \geq 0$ and $\arg(\lambda) = \pi$ for $\lambda < 0$.

Substitution of the partial frequencies $\omega_1, \omega_2, \ldots, \omega_n$ and phases $\theta_1, \theta_2, \ldots, \theta_n$ into Eqs. (4.4) and (4.5) provides a system of $2n$ nonlinear equations in $2n+1$ unknowns (with regard for λ). To close the equation system, we turn to the initial and boundary conditions (see Property 3) which enables us to obtain the phase-frequency equation

$$\chi(\omega_1, \omega_2, \ldots, \omega_n, \theta_1, \theta_2, \ldots, \theta_n) = 0 \tag{4.6}$$

relating the partial parameters. Its derivation is explained in the Appendix. In particular, for the first-order blocks (integrator and aperiodic block), it has a very simple form of $\theta_1 = 0$. For the ideal oscillatory block, we get the aforementioned invariant relationship $\omega_1 \cot(\theta_1) - \omega_2 \cot(\theta_2) = 0$. The necessity to consider the phase-frequency equation reflects the specific aspects of the finite time interval. Therefore, a closed system of Eqs. (4.4)–(4.6) was obtained for determining the singular values and their corresponding singular functions of the convolution operator. It can be solved either analytically, if possible, or numerically.

Determination of the characteristic equation for the singular values of the convolution operator was among the problems formulated at the beginning of this paper. The equation system obtained suggests a three-step procedure of its generation.

Step 1. By using the amplitude balance equation system (4.4), express the partial frequencies as the functions of λ: $\omega_1(\lambda), \omega_2(\lambda), \ldots, \omega_n(\lambda)$.

Step 2. Substitute the resulting expressions for the partial frequencies in the phase balance Eqs. (4.5) and determine the phase relations $\theta_1(\lambda)$, $\theta_2(\lambda), \ldots, \theta_n(\lambda)$.

Step 3. Substitute the resulting expressions for the partial frequencies and phases into the phase-frequency Eq. (4.6).

As the result, we obtain the required characteristic equation which is transcendental and has a countable set of roots.

The first stage of this procedure is interpreted graphically in Fig. 2 depicting by way of example the GFC of a fourth-order system which has two resonance frequencies and corresponds to the first term of Eq. (4.4). The second term is represented by the horizontal line. Four partial frequencies correspond to their intersection points in the neighborhoods of the system resonance frequencies. If they are pairwise close, then beatings become apparent in the singular function. The hyperbolic

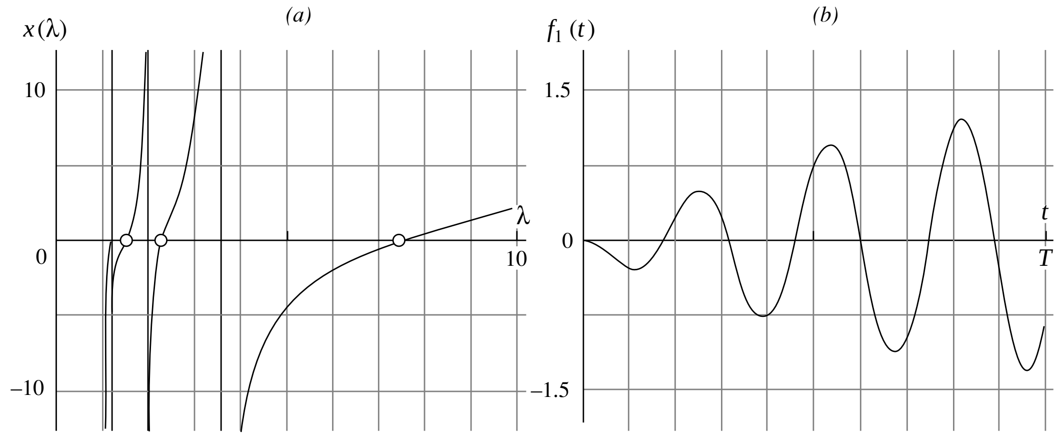

Fig. 3. Positive roots of the characteristic equation of the conservative block (a) and its main singular function (b).

components of the singular functions arise for smaller absolute values of λ where the number of the intersection points is smaller than the system order. The above procedure is illustrated by two examples.

Example 1. We begin with the conservative system which best yields to the frequency analysis and determine the characteristic equation and the singular functions of the convolution operator for the block with the transfer function $\mathrm{Q}\,(p) = 1/(p^2+1)$.

The amplitude balance Eq. (4.4) is as follows:

$$1/(1-\omega^2)^2 + \lambda^2 = 0.$$

Hence, the two partial frequencies obey the expressions $\omega_1(\lambda) = \sqrt{1-\dfrac{1}{|\lambda|}}$ and $\omega_2(\lambda) = \sqrt{1+\dfrac{1}{|\lambda|}}$.

The phase balance Eq. (4.5) is put down with regard for the earlier equality $\psi(\omega,\theta) = \pi - \omega\,\mathrm{T} - 2\theta$ (PFC of the flip operator):

$$\varphi(\omega) + \pi - \omega\,\mathrm{T} - 2\theta = \arg(\lambda),$$

where $\varphi(\omega) = \arg(1-\omega)$ is the PFC of the oscillatory block which is representable as $\varphi(\omega_1) = 0$, $\varphi(\omega_2) = \pi$. Here, ω_1 and ω_2 are the partial frequencies in the neighborhood of the special point $\omega_0 = 1$ that allow us to represent the phases as $\theta_1 = (\pi - \omega_1\,\mathrm{T})/2$ and $\theta_2 = -\omega_2\,\mathrm{T}/2$ (if $\lambda > 0$) and, otherwise, $\theta_1 = -\omega_1\,\mathrm{T}/2$ and $\theta_2 = (\pi - \omega_2\,\mathrm{T})/2$ (if $\lambda < 0$).

It remains to use the phase-frequency Eq. (4.6)

$$\omega_1 \cot(\theta_1) - \omega_2 \cot(\theta_2) = 0.$$

By substituting into it the partial frequencies and phases, we get for the positive λ the transcendental characteristic equation

$$\sqrt{1-\frac{1}{\lambda}}\cot\left(\frac{\mathrm{T}}{2}\sqrt{1-\frac{1}{\lambda}}\right) + \sqrt{1+\frac{1}{\lambda}}\cot\left(\frac{\mathrm{T}}{2}\sqrt{1+\frac{1}{\lambda}}\right) = 0.$$

Its graphic solution for $\mathrm{T} = \dfrac{10\,\pi}{\sqrt{2}} \approx 22.3$ is shown in Fig. 3a as the points of intersection of the left function and the abscissa. The spectral points condense as the origin approaches (the figure shows the three first points of the spectrum). For the interval under consideration, the values of roots at these points are 7.43 (the main singular value), then 2.31 and 1.54. Their corresponding singular functions are of oscillatory nature:

$$f(\lambda, t) = \sin(\omega_1(\lambda)t + \theta_1(\lambda))\sin(\theta_2(\lambda)) - \sin(\omega_2(\lambda)t + \theta_2(\lambda))\sin(\theta_1(\lambda)).$$

The main function is shown in Fig. 3b. The force swinging the oscillator (pendulum, bell) according to this law must coincide with the right singular function which is the mirror reflection of the presented graph relative to the middle of the interval. It causes the maximum-energy response over the given time interval.

Example 2. From the point of view of the frequency approach, the first-order blocks are less convenient for analysis. Let us consider the block with the transfer function $\mathrm{Q}(p) = 1/(p + b)$. For $b = 0$, this model describes the integrator for $b > 0$, the stable aperiodic block, and for $b < 0$, the unstable aperiodic block.

First, we rely on Property 1 and formula (3.6) to localize the partial spectrum of the singular functions. In the case at hand, for $p = a + j\omega$ we get

$$R(p) = \mathrm{Q}(p)\mathrm{Q}(-p) = 1/(b^2 - p^2) = 1/(b^2 - a^2 + \omega^2 - 2ja\omega).$$

According to the first condition $\mathrm{Im}\,(R(p)) = 0$ of (3.6), we get $a\omega = 0$. Hence, either $a = 0$ or $\omega = 0$, that is, the partial spectrum is concentrated on the real and imaginary axes of the complex plane. In the first case, the singular function is the hyperbolic dependence of the form $f(t) = \sinh(at + \theta)$, in the second, the sinusoid $f(t) = \sin(\omega t + \theta)$.

According to the second condition $\mathrm{Re}\,(R(p)) \geq 0$ of (3.6), we get $b^2 + \omega^2 > 0$. Hence, the admissible domain of the real axis is bounded by the segment $-b \leq a \leq b$. For the integrator with $b = 0$, this segment, in particular, can degenerate into a point. Therefore, the integrator cannot have the hyperbolic singular functions.

After spectrum localization, we consider the balance equations. The ordinary system GFC and the hyperbolic GFC (underlined below) obeying Eq. (4.1) provide two equations of the amplitude balance for the partial frequencies:

$$|\lambda| = \mathrm{A}(\omega) = \sqrt{\frac{1}{b^2 + \omega^2}}, \quad \text{hence,} \quad \omega = \sqrt{1/\lambda^2 - b^2}, \quad \text{and}$$
$$|\lambda| = \underline{\mathrm{A}}(a) = \sqrt{\frac{1}{b^2 - a^2}}, \quad \text{hence,} \quad a = \sqrt{b^2 - 1/\lambda^2}.$$

The second equation is valid for the eigenvalues λ exceeding the static gain of the block.

The ordinary PFC of the aperiodic block has the form $\varphi(\omega) = -\arctan(\omega/b)$. To make the picture comprehensive, one could introduce the hyperbolic PFC $\varphi(a) = -\mathrm{arctanh}\,(a/b)$ disregarding the change of sign of the harmonic for $b < 0$. It follows from the phase-frequency equation (zero initial conditions) that $\theta = 0$. Therefore, the PFCs of the flip operator for the periodic and aperiodic modes are, respectively, $\psi(\omega) = \pi - \omega\,\mathrm{T}$ and $\psi(a) = -a\,\mathrm{T}$, the latter formula disregarding the change of sign of the harmonic. The block delaying for the time T has a similar characteristic. Hence, we get the pair of phase balance equations for the periodic and aperiodic modes: $-\arctan(\omega/b) + \pi - \omega\mathrm{T} = \arg(\lambda)$ and $-\mathrm{arctanh}\,(a/b) - a\,\mathrm{T} = 0$. The hyperbolic antitangent $\mathrm{arctanh}\,(a/b)$ for $a < 0$ intersects the linear function $-a\,\mathrm{T}$ only at one point. Consequently, there can be only one hyperbolic harmonic of the unstable block for $-b\,\mathrm{T} > 1$. It is the main singular function of the unstable aperiodic block which corresponds to the maximum singular value. The rest of the singular functions are

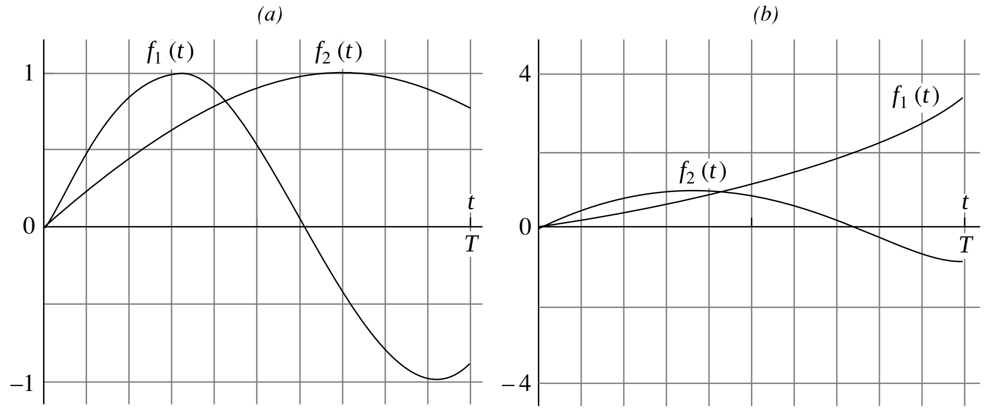

Fig. 4. Two first singular functions of the stable (a) and unstable (b) aperiodic blocks.

the ordinary sinusoidal ones whose frequencies can be established using the first phase-balance equation.

For positive λ, the characteristic equations for the periodic and aperiodic modes are as follows:

$$-\arctan\left(\sqrt{1-\frac{1}{\lambda^2 b^2}}\right)+\pi-\mathrm{T}\sqrt{1-\frac{1}{\lambda^2 b^2}}=0 \quad \text{and}$$
$$-\operatorname{arctanh}\left(\sqrt{\frac{1}{\lambda^2 b^2}-1}\right)-\mathrm{T}\sqrt{\frac{1}{\lambda^2 b^2}-1}=0.$$

The frequencies $\omega_i = \sqrt{1-\frac{1}{\lambda_i^2 b^2}}$ and $a = \sqrt{\frac{1}{\lambda^2 b^2}-1}$ and the singular functions $f_1 = \sin(\omega_1 t)$ and $f_2 = \sin(\omega_2 t), \ldots$ for the stable block and $f_1 = \sinh(at)$ and $f_2 = \sin(\omega_2 t), \ldots$ for the unstable one for $-b\,\mathrm{T} > 1$ correspond to the roots of these equations. Figure 4 shows examples of such functions for $b = \pm 1$ and $\mathrm{T} = 2$. In the critical case, the hyperbolic dependence is replaced for $-b\,\mathrm{T} = 1$ by the linear dependence $f_1 = t$ and then by the sinusoidal dependence $f_1 = \sin(\omega_1 t)$ with the decrease of T. For small values of T, the characteristics, including the singular functions, of both stable and unstable aperiodic blocks in that way approach the integrator.

5. GRAPHO-ANALYTICAL STUDY OF THE SPECTRAL BALANCE EQUATIONS

The frequency theory of the automatic control systems was created prior to the advent of computers. It worked out an efficient grapho-analytical method for determining stability of the dynamic systems which underlies, in particular, the popular criteria of Nyquist, Popov, and others. In what follows, we extend this approach to obtain the spectral characteristics of the convolution operator over the bounded time interval.

The frequency approach offers a transparent interpretation of the partial frequencies as the points of GFC taken at the discrete values of the argument, that is, the frequencies satisfying the phase constraints. It is important not only as a method providing a particular result, but also because it reveals the profile of a particular discrete spectrum vs. variations of the interval T and the characteristics of the dynamic blocks such as time constants, damping coefficients, and so on. It is indicative of the nonuniform nature of distribution of the partial frequencies.

To demonstrate the main regularities, let us consider the frequency characteristics of the integrator over the time interval $\mathrm{T} = 5$. Figure 5a shows its GFC $\mathrm{A}(\omega) = 1/\omega$ (above the frequency axis)

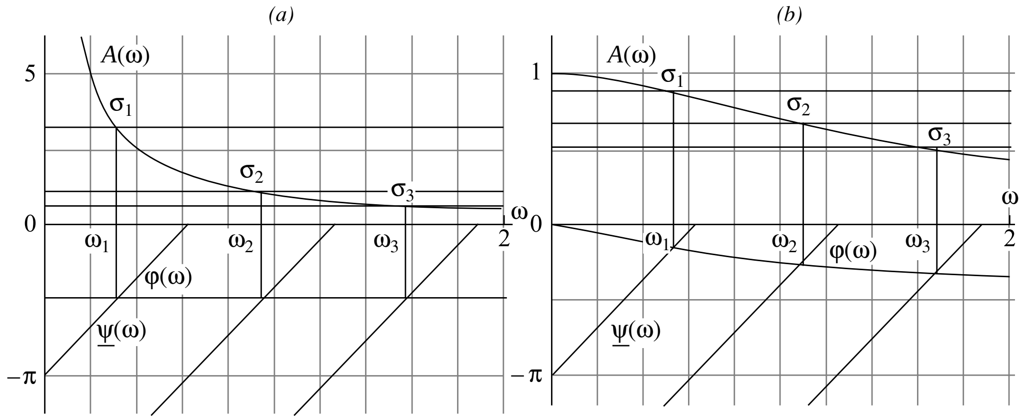

Fig. 5. Frequency characteristics of the integrator (a) and aperiodic block (b) over the finite time interval.

and PFC $\varphi(\omega) = -\pi/2$ (below the frequency axis, the axes of both characteristics are in line, but the up and down scales are different), as well as the shifted PFC $\underline{\psi}(\omega) = \omega\,\mathrm{T} - \pi$ of the flip operator which is inverted in sign for convenience of taking into account for phase balance $\varphi(\omega) = \underline{\psi}(\omega)$. We recall that the phase-frequency equation provides here a trivial solution in the phase shift $\theta = 0$. With regard for the sinusoidal form of the singular functions, the lines of frequency characteristics of the flip operator are multiplied with the interval 2π. Therefore, from the intersection points of the frequency characteristics of the system $\varphi(\omega)$ and the flip operator $\underline{\psi}(\omega)$ one establishes the frequency ω_1 of the main singular function corresponding to the maximum singular value σ_1, as well as the frequencies and singular values of the remaining singular functions $f_k(t) = \sin(\omega_k t)$. Figure 5b shows for comparison similar frequency characteristics of the aperiodic block. In this case, the intersection points of the curves $\varphi(\omega)$ and $\underline{\psi}(\omega)$ lies irregularly on the frequency axis.

The above analysis shows that the greater T, the denser the points of the discrete spectrum on the continuous GFC. In the domain of high frequencies (for small T), the spectral characteristics of various dynamic systems approach each other. The partial frequencies of the oscillatory systems are concentrated about the resonance peaks of the GFC and make up close pairs. A similar splitting of spectrum is well known in physics.

At high frequencies corresponding to smaller singular values, the resonance peaks do not define any more the nature of the singular functions; they more and more approach the characteristics of the n-fold integrator. Therefore, their study is of special interest. Hyperbolic components appear in the partial spectra of the "minor" singular functions of the minimum-phase systems of the second and higher orders. Their analysis by the frequency methods presents certain difficulties which can be avoided by considering simpler singular functions because any of them (like the weight function) describes the system uniquely.

6. CONCLUSIONS

Analysis of the behavior of dynamic systems over the finite time interval is important to many applications. The classical theory of linear systems is largely oriented to the infinite or semiinfinite time intervals. This concerns the apparatus of frequency characteristics, Laplace transform, Kalman filters, stability analysis, and other domains where many useful elegant results were obtained. In practice, however, this approach can be used only for the systems whose time of operation is much greater than the duration of the transients. But the physical systems often operate over finite time intervals commensurable with the time of system transients, thus making many classical results invalid or senseless. For example, this refers to the issues of system stability over the finite time interval and the role played by the sinusoidal harmonic signals in the frequency analysis.

Therefore, it seems important to study, first, the results and propositions of the classical theory that retain their validity—possibly, with some modifications—over finite time intervals and, second, the new effects and properties that arise here. In this sense, one can declare the need for developing the finite theory of linear systems. The paper studied one its aspect concerned with the spectral characteristics of the convolution operator over the finite time interval. A procedure for determining its singular values and singular functions was developed, and their properties were studied. For nonbounded increase of the time interval, the singular functions become sinusoidal, but over the finite interval they are of quite different nature and represent the polyharmonic signals with different partial frequencies. A closed system of algebraic equations was obtained for determining the partial frequencies and the singular values of the convolution operator.

It is only natural that the possibilities of the analytic methods for the finite time interval are limited. For example, the characteristic equation of the convolution operator is transcendental, and its roots cannot be established analytically even for the first-order systems. In this connection, the paper suggested numerical and grapho-analytical procedures for determining the spectral characteristics that share traits with the frequency design of the automatic control systems by GFC and PFC.

The applied value of the results obtained lies in that the singular functions of the convolution operator have some extremal features and can prove useful in the problems of optimal control. It seems promising to expand the results obtained to a wider range of systems, in particular, to the multidimensional and non minimum-phase dynamic systems.

APPENDIX

Proof of Theorem 1 (on mirror symmetry). According to Definition 1, the right singular functions of the convolution operator g_i satisfy $S^*Sg_i = \sigma_i^2 g_i$. We use $S^* = FSF$ to rearrange them in

$$FSFSg_i = \sigma_i^2 g_i \quad \text{or} \quad H^2 g_i = \sigma_i^2 g_i, \tag{A.1}$$

where $FS = H$ is the symmetrical operator obtained by polar factorization of the convolution operator.

Let q_i be the eigenfunction of the operator H corresponding to its eigenvalue λ_i; then, the following equalities are valid:

$$Hq_i = \lambda_i q_i. \tag{A.2}$$

Comparison of (A.1) and (A.2) suggests that the singular functions of the operators S^*S and H coincide, $g_i = q_i$, and their eigenvalues are equal to the sign $\lambda_i = \pm\sigma_i$. Consequently,

$$Hg_i = \lambda_i g_i, \quad FSg_i = \lambda_i g_i, \quad Sg_i = \lambda_i F f_i.$$

Hence, with regard for the second relation in (2.7b) we obtain that $\sigma_i f_i(t) = \lambda_i g_i(\mathrm{T} - t)$ or $f_i(t) = \pm g_i(\mathrm{T} - t)$, which is what we set out to prove.

Proof of Theorem 2 (on partial spectrum). For a system with the transfer function $\mathrm{Q}\,(p) = \dfrac{b(p)}{a(p)}$, the left and right singular functions satisfy the following relations:

$$\mathrm{Q}(p)\mathrm{Q}(-p)f(p) = \sigma^2 f(p), \quad \mathrm{Q}(-p)\mathrm{Q}(p)g(p) = \sigma^2 g(p)$$

to which the same characteristic equation

$$\sigma^2 a(p)a(-p) + b(p)b(-p) = 0 \tag{A.3}$$

for the partial spectrum of singular functions corresponds. Since it involves only the even degrees of p, its root locus on the complex plane is characterized by the central symmetry about the origin. The total number of roots is $2n$; their pairwise aggregation provides n harmonics involved in the right and left singular functions.

Circulatory harmonics $h_i(t) = a_i \sin(\omega t) + b_i \cos(\omega t)$, hyperbolic harmonics $h_j(t) = a_j \sinh(\omega t) + b_j \cosh(\omega t)$, and mixed harmonics $h_s(t) = \sinh(\omega_1 t)[a_s \sin(\omega_2 t) + b_s \cos(\omega_1 t)]$ and $h_c(t) = \cosh(\omega_1 t)[a_c \sin(\omega_2 t) + b_c \cos(\omega_2 t)]$ correspond, respectively, to pairs of the purely imaginary roots, the purely real roots, and quadruples (quartettes) of the double-symmetrical complex roots. The variational methods based on the Lagrange multipliers follow a longer way to the same result.

Proof of the Corollary to Theorem 2. The polynomial pairs $(\pm a(p), \pm b(p))$ of the minimum-phase systems have no roots in common. Therefore, the roots of the polynomials $\pm a(p)$ and $\pm b(p)$, that is, the zeros and poles of the original and conjugate systems cannot be the roots of Eq. (A.3). Consequently, the harmonics $h_i(t)$ do not coincide with the modal components of the pulse weight functions of the original and conjugate systems.

Additionally, the singular functions cannot comprise isolated exponential terms, because by virtue of root symmetry a term e^{-at} will correspond to each terms of the form e^{at}; together they make up a hyperbolic function. We note that with the availability of the multiple roots of Eq. (A.3), components comprising polynomial multipliers will occur in the singular function.

The polynomials $a(p)$ and $b(-p)$ of the nonminimum-phase systems may have common roots. Then, the singular function includes the corresponding modal components of the system. Any nonminimum-phase system is known to be representable as a series connection of the minimum-phase system and the phase shifter. The aforementioned modal components correspond to its poles.

Proof of Property 1 (spectrum localization). Both the right and left singular functions of the convolution operator satisfy the relation $\mathrm{Q}(p)\,\mathrm{Q}(-p)f(p) = \sigma^2 f(p)$ from which we pass to the operator equality $R(p) - \sigma^2 = 0$, where $R(p) = \mathrm{Q}(p)\,\mathrm{Q}(-p)$. By taking into account reality of σ, we put to zero the real and imaginary parts of the complex expression $R(p) - \sigma^2$:

$$\mathrm{Im}\,(R(p)) = 0, \quad \mathrm{Re}\,(R(p)) = \sigma^2,$$

which proves (3.6). The first of these relations implies that there is no phase shift in the singular function passing through the series connection of the direct and conjugate systems. The second one means that all partial components of the singular function have the same gain.

Proof of Property 2 (boundary conditions). According to Theorem 1,

$$\lambda f(t) = \int_0^t q(t-\tau) f(\mathrm{T}-\tau)\,d\tau, \tag{A.4}$$

where $q(t)$ is the pulse weight function of the system. By assuming that $t = 0$, we get $f(0) = 0$, that is, the first of the relations (3.7). By differentiating (A.4) with respect to time, we obtain

$$\lambda \dot{f}(t) = q(0)f(\mathrm{T}-t) + \int_0^t \dot{q}(t-\tau) f(\mathrm{T}-\tau)\,d\tau$$

which after substitution of $t = 0$ provides $\lambda \dot{f}(0) = q(0)f(\mathrm{T})$, that is, the second of the relations of (3.7). Repeated differentiation and integration by parts lead to the third relation of (3.7), and so on.

There exists a matrix variant of the proof which is based on considering the equalities $y = \mathrm{C}x$, $\dot{y} = \mathrm{CA}\,x + \mathrm{CB}\,u$, and others obtained for the input $u = f(t)$ by successive differentiation. We note that the listed boundary conditions were put down with regard for the initial system conditions.

Derivation of the phase-frequency equation. If the relative order of the system is n, that is, $\mathrm{Q}\,(p) = b_0/a(p)$, then the first $n-1$ Markov parameters will be zero. In this case, the relations for the boundary conditions are simplified and, in particular, for the four first conditions

$$\lambda \begin{pmatrix} f(0) \\ \dot{f}(0) \\ \ddot{f}(0) \\ \dddot{f}(0) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ q(0) & 0 & 0 & 0 \\ \dot{q}(0) & q(0) & 0 & 0 \\ \ddot{q}(0) & \dot{q}(0) & q(0) & 0 \end{pmatrix} \begin{pmatrix} f(\mathrm{T}) \\ -\dot{f}(\mathrm{T}) \\ \ddot{f}(\mathrm{T}) \\ -\dddot{f}(\mathrm{T}) \end{pmatrix}$$

they assume the form

$$\begin{pmatrix} f(0) \\ \dot{f}(0) \\ \ddot{f}(0) \\ \dddot{f}(0) \end{pmatrix} = 0.$$

Here, the frequencies and phases of the partial components of any sinusoidal singular function

$$f(t) = a_1 \sin(\omega_1 t + \theta_1) + \ldots + a_4 \sin(\omega_4 t + \theta_4)$$

should satisfy the determinantal relations like

$$\begin{vmatrix} \sin\theta_1 & \sin\theta_2 & \sin\theta_3 & \sin\theta_4 \\ \omega_1\cos\theta_1 & \omega_2\cos\theta_2 & \omega_3\cos\theta_3 & \omega_4\cos\theta_4 \\ \omega_1^2\sin\theta_1 & \omega_2^2\sin\theta_2 & \omega_3^2\sin\theta_3 & \omega_4^2\sin\theta_4 \\ \omega_1^3\cos\theta_1 & \omega_2^3\cos\theta_2 & \omega_3^3\cos\theta_3 & \omega_4^3\cos\theta_4 \end{vmatrix} = 0,$$

which is invariant to the number of the singular function. It is the phase-frequency Eq. (4.6)

$$\chi(\omega_1, \omega_2, \ldots, \omega_n, \theta_1, \theta_2, \ldots, \theta_n) = 0$$

relating the partial parameters. For $n = 1$, we obtain, in particular, $\sin\theta = 0$. For $n = 2$, we get

$$\begin{vmatrix} \sin\theta_1 & \sin\theta_2 \\ \omega_1\cos\theta_1 & \omega_2\cos\theta_2 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} \tan\theta_1 & \tan\theta_2 \\ \omega_1 & \omega_2 \end{vmatrix} = 0,$$

whence the above expression $\omega_1/\tan(\theta_1) = \omega_2/\tan(\theta_2)$ follows which is valid for any oscillatory second-order block with a constant in the numerator. For $n = 3$, the phase-frequency equation is as follows:

$$\begin{vmatrix} 1 & 1 & 1 \\ \omega_1\cot\theta_1 & \omega_2\cot\theta_2 & \omega_3\cot\theta_3 \\ \omega_1^2 & \omega_2^2 & \omega_3^2 \end{vmatrix} = 0.$$

This procedure of deriving the phase-frequency equation is valid in a more general case of arbitrary Markov parameters and forms of the partial components.

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