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Generalized Scarpis Methods for Hadamard Matrix Calculation

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Abstract

This paper discusses generalizations of the Scarpis methods, interpreted as the Kronecker product of quasiorthogonal (Cretan) matrices of even (Hadamard matrices) and odd (Hadamard matrix cores, Mersenne, and Seidel matrices) orders, allowing, after correction, for calculation of Hadamard matrices of higher orders. The complexity of correction depends on both the symmetry (asymmetry), and the place of factor matrices in the product. The possibility to calculate a product of factor matrices with orders not equal to odd prime numbers, is shown. New realizations of the direct Scarpis method are studied and the inversed Scarpis method is presented. The border is added to a normalized matrix in the direct Scarpis method, and it is removed in the inversed method. Modified methods of multiplication of Cretan matrices – local maximum determinant matrices – generalize the previously known methods of their calculation.

Mathematics Subject Classification: 05B20; 20B20

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1 Introduction

Matrices with orthogonal columns (rows) are widely used in data processing and transformation. These are, for instance, Hadamard matrices **H** [1] of orders 1, 2 and n=4t, where t is an integer, with entries {1, –1}. For them, the equation $\mathbf{H}_n^T \mathbf{H}_n = n\,\mathbf{I}$ is true, where **I** is a unity matrix.

Decreasing the order n of a normal Hadamard matrix by 1 by way of removing the borders (a row and a column with the unity elements) gives a so-called core, which, in its turn, can be orthogonalized by way of changing the value of one of its entries. Matrixes of the n=4t–1 order obtained using this way care called Mersenne matrices [2, 3].

Paper [3] provides an algorithm for building Mersenne martices (**M**) of orders, equal to Mersenne numbers $n=2^k-1$, where k is an integer, with entries {1, –b}. For Mersenne martices $\mathbf{M}_n^T\mathbf{M}_n = \omega\,\mathbf{I}$ is true, where ω is the matrix weight. The value of entry $b = \frac{t}{t+\sqrt{t}}$ is a systematizing property of these matrices. Paper [4] proves that all Mersenne matrices can be extended to all values of the real axis n=4t–1, where t is an integer.

Fermat matrixes (**F**) available for orders equal to Fermat numbers $2^{2^k}+1$, have a similar property. A range of these matrices, for which $\mathbf{F}_n^T\mathbf{F}_n = \omega\,\mathbf{I}$ is true, can be extended for values $n=4t^2+1$, where t is an integer [2, 5].

Matrices of even orders n=4t–2 are easy to obtain by doubling the Mersenne matrices. Such matrices, as well as similar to them Euler matrices (**E**), exist even where n–1 cannot be represented as the sum of squares of two numbers. The entries of these matrixes may have irrational values [4, 5].

Besides the theoretical interest in the existence theorems and matrix analogues of basic numerical sequences, there is a practical interest in obtaining the as many quasi orthogonal matrices as possible [5], including Hadamard matrices.

2 Kronecker Matrix Product

The theory of orthogonal matrixes gives the Kronecker product of **A**⊗**B** matrices **A** and **B** a role of a generator of matrices of larger orders. This product can be described as the insertion of matrix **B** in places of elements in matrix **A** with multiplication by these entries' values, like this:

$$\mathbf{A}\otimes\mathbf{B} = \begin{pmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \ldots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}\mathbf{B} & a_{n2}\mathbf{B} & \ldots & a_{nn}\mathbf{B} \end{pmatrix}.$$

Orthogonality is an invariant in this product in terms of possible change of factors: no matter which orthogonal by columns (rows) matrices we choose, the result will be an quasiorthogonal matrix.

If the elements (levels) of factors **A** and **B** are different from 1 and –1, then the Kronecker product significantly increases their number. The increase of the number of levels is undesirable in the case where matrices are divided into equivalence classes depending on the quantity and values of their entries. This applies to generalizations of Hadamard matrices discussed above, also known as *Cretan matrices* [6], of even and odd orders.

For binary Cretan matrices the Kronecker product describes a structure that can be easily reduced back to a binary matrix by way of *correcting its entries*. This was noticed by Scarpis [7], who used matrices of closed even (Hadamard matrix) and odd (its core) orders as factors. The correction was difficult. Working on simplification of the Scarpis method, Paley [8] excluded the matrix of odd orders from consideration. He used the fact that the three-level Belevitch matrices (conference matrices) [9] of even orders are either asymmetric (skew symmetric), or symmetric, which reduces the correction formula.

These two Paley constructions can be summarized as follows.

Asymmetric Belevitch matrix (**C**) of the n=4t order, where t is an natural number, allows to get a Hadamard matrix by $\mathbf{H}_n=\mathbf{C}_n+\mathbf{I}$ correction, since the orthogonality of columns is an invariant, not depending on the values of the diagonal entry.

Kronecker product of the symmetric Belevitch matrix of even order $n=4t-2$ and the Hadamard matrix $\mathbf{H}_2=\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ results in Hadamard matrix of the doubled order 2n after similar correction of zero entries **C** in the product formula

$$\mathbf{H}_n=\begin{pmatrix}\mathbf{C}_n+\mathbf{I} & \mathbf{C}_n-\mathbf{I}\\ \mathbf{C}_n-\mathbf{I} & -\mathbf{C}_n-\mathbf{I}\end{pmatrix}.$$

Corrections to the Kronecker product of Cretan matrixes [2, 4, 5] can also be simplified if taken in symmetric or asymmetric (near to skew symmetric) form.

3 Scarpis Product

The described product is a historical misconception. About a hundred years ago Scarpis inserted a normalized Hadamard matrix of the n=4t order into its core of the 4t–1 order and noticed that the orthogonalization of columns (and rows) in such an insertion is possible if the Hadamard matrix block is cyclically shifted proportionally to the distance of the replaced entry from the upper left corner of the matrix.

We improved this method, making it significantly shorter than the original one [7], by way of implementing another order of permutations with the use of normal form of the Hadamard matrix and its core. The improved method can be written in the form of one formula of the Kronecker product with shifting correction (Scarpis product ×):

$$\mathbf{M}\times\mathbf{H}=\begin{pmatrix} \begin{pmatrix} -1 & m_{11}\mathbf{e}^{\mathrm{T}} \\ m_{11}\mathbf{e} & \mathbf{M} \end{pmatrix} & \begin{pmatrix} -1 & m_{12}\mathbf{e}^{\mathrm{T}} \\ m_{12}\mathbf{e} & \mathbf{M} \end{pmatrix} & \cdots & \begin{pmatrix} -1 & m_{1(n-1)}\mathbf{e}^{\mathrm{T}} \\ m_{1(n-1)}\mathbf{e} & \mathbf{M} \end{pmatrix} \\ \begin{pmatrix} -1 & m_{21}\mathbf{e}^{\mathrm{T}} \\ m_{21}\mathbf{e} & \mathbf{M} \end{pmatrix} & \begin{pmatrix} -1 & m_{22}\mathbf{e}^{\mathrm{T}} \\ m_{22}\mathbf{e} & \mathbf{TM} \end{pmatrix} & \cdots & \begin{pmatrix} -1 & m_{2(n-1)}\mathbf{e}^{\mathrm{T}} \\ m_{2(n-1)}\mathbf{e} & \mathbf{T}^{n-2}\mathbf{M} \end{pmatrix} \\ \cdots & \cdots & \ddots & \cdots \\ \begin{pmatrix} -1 & m_{(n-1)1}\mathbf{e}^{\mathrm{T}} \\ m_{(n-1)1}\mathbf{e} & \mathbf{M} \end{pmatrix} & \begin{pmatrix} -1 & m_{(n-1)2}\mathbf{e}^{\mathrm{T}} \\ m_{(n-1)2}\mathbf{e} & \mathbf{T}^{n-2}\mathbf{M} \end{pmatrix} & \cdots & \begin{pmatrix} -1 & m_{(n-1)(n-1)}\mathbf{e}^{\mathrm{T}} \\ m_{(n-1)(n-1)}\mathbf{e} & \mathbf{T}^{(n-2)(n-2)}\mathbf{M} \end{pmatrix} \end{pmatrix},$$

where $\mathbf{H}=\begin{pmatrix} -1 & \mathbf{e}^{\mathrm{T}} \\ \mathbf{e} & \mathbf{M} \end{pmatrix}$ is a Hadamard matrix, **M**=–core(**H**) is a Mersenne matrix rounded to integer values, **e** is a vector of unit cells of the border, and **T** is a matrix of cyclic shifting. Each block shifts by (i–1)(j–1).

Both rows or columns can be shifted, since the procedure is symmetric. The sign of a replaced entry affects only the signs of border entries.

In his paper [1] Hadamard provided matrixes of orders 12 and 20. Using his method, Scarpis calculated a new Hadamard matrix of a high composite order 7·8=56. The method is indifferent to the type of matrix **M**, but it can be used only for simple orders n=4t–1. In the general case the method is faulty – a more complex correction is required to get the result.

The misconception is that the Kronecker product with corrections, as we found out, can be applied to matrices of any order, not only to those with prime numbers as orders. This was not previously noticed, because after Paley these products were not paid much attention to, and the Scarpis method with shifts was almost forgotten. However, the complexity of correction of the Kronecker product depends on the type of symmetry of factors. Symmetric and asymmetric forms of matrixes were studied in the past and are studied today [10–14].

4 Generalized Scarpis Product

New generalized forms of the method are presented in this paper for the first time ever. It was found that when the reversed sequence of factor matrixes is used in the Kronecker product, the shifting of rows in an *asymmetric* matrix **M** (**M**–**I** it is skew) is not required when inserting it into matrix $\mathbf{H}=\begin{pmatrix} 1 & -\mathbf{e}^{\mathrm{T}} \\ \mathbf{e} & \mathbf{M} \end{pmatrix}$.

This property is absolute, it's indifferent to the requirement that the order of the **M** matrix should be a prime number, but depends on its form. When the symmetry of factors is restricted, the *correction* comes down to entries {1, –*b*} of *diagonal* matrices **M** becoming positive. Being rounded, this matrix equals **J** – a matrix with entries 1, with the order of matrix **M**.

The generalized Scarpis product (method) can be written as follows:

$$[\mathbf{H}\otimes\mathbf{M}]= \mathbf{I}\otimes\mathbf{J} + (\mathbf{H}–\mathbf{I})\otimes\mathbf{M},$$

where **H** is an asymmetric matrix, **I** is an identity matrix of the same order, and **M**=–core(**H**) is the Mersenne matrix rounded to integers.

Fig. 1 depicts the portraits of Hadamard matrixes, obtained using the Scarpis method for two realizations described above – with shifting of rows of the **M** block, and with correction in the form of **J** on the diagonal. The figure shows an entry with value –1 in red color, and with value 1 in white. The benefit from transposition of factors is significant.

Our more generalized formula of the Scarpis product is true for **M** of any integer orders 4*t*–1: for prime numbers (3, 7, 11, etc.); prime powers (for instance, $3^3=3\cdot3\cdot3=27$); products of close integers $3\cdot5=15$, $7\cdot9=35$, etc. Using this way, for example, the Hadamard matrix of the 756 order was calculated, which is impossible using the conventional Scarpis method.

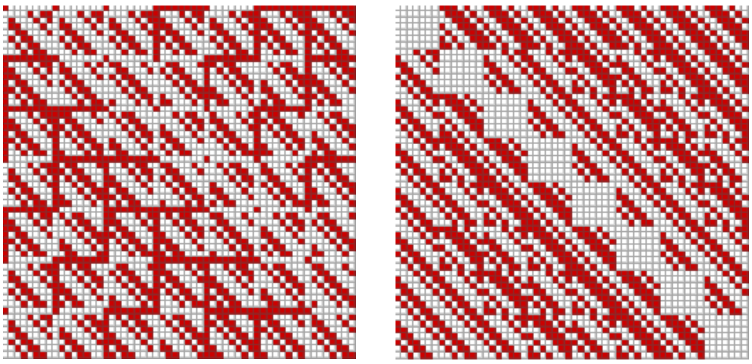

Fig. 1. Portraits of Matrixes $\mathbf{H}_{56}$, obtained using the generalized forms of the Scarpis method

5 Inversed Scarpis Method

In the original Scarpis method, a border is added to the **M** matrix, then a Hadamard matrix **H** is built with it to be inserted into matrix **M** with cyclic shifting of the internal block in the calculation of **M**×**H**. It should be noted that the border works as a compensator, which can be added or removed. If the border is removed, it is possible to calculate Hadamard matrixes of composite orders different to those discussed above.

We propose the new *inversed* Scarpis method.

Let us take matrix **M** in the bicyclic form with a border of $2q+1$ order and remove the border as an unnecessary compensator. Now we have

$$\mathbf{M}=\begin{pmatrix}1 & \mathbf{e}^{\mathrm{T}} & -\mathbf{e}^{\mathrm{T}}\\ -\mathbf{e} & \mathbf{A} & \mathbf{B}\\ \mathbf{e} & -\mathbf{B}^{\mathrm{T}} & \mathbf{A}^{\mathrm{T}}\end{pmatrix},\quad \mathbf{H}=\begin{pmatrix}\mathbf{A}\times[\mathbf{A}] & \mathbf{B}\times[\mathbf{B}]\\ -(\mathbf{B}\times[\mathbf{B}])^{\mathrm{T}} & (\mathbf{A}\times[\mathbf{A}])^{\mathrm{T}}\end{pmatrix}.$$

The process of calculation of this product is as follows:

- Allocate two blocks with odd orders **A** and **B** in the bicyclic core of matrix **M** of even order $2q$;

- Build their formal extensions $[\mathbf{A}]=\begin{pmatrix}-1 & \mathbf{e}^{\mathrm{T}}\\ \mathbf{e} & \mathbf{A}\end{pmatrix},[\mathbf{B}]=\begin{pmatrix}-1 & \mathbf{e}^{\mathrm{T}}\\ \mathbf{e} & \mathbf{B}\end{pmatrix}$;

- Use the given Scarpis formula to quadrate the internal blocks.

The result of this calculation is the extended matrix **H** of order $2q(q+1)$.
The portrait of matrix $\mathbf{H}_{60}$, obtained from a bicyclic matrix $\mathbf{M}_{11}$ with blocks sized $q=5$, is shown on Fig. 2. Another modification of the Scarpis method that we propose consists in the following: for matrix $\mathbf{H}_n$ we take a cyclic matrix $\mathbf{M}_{n+3}$ and they are multiplied as $[\mathbf{H}\otimes\mathbf{M}]$ with correction. With difference = 3 between the values of factors' orders, the diagonal compensator is a matrix with entries –1 on the antidiagonal. For $\mathbf{H}_8$ and $\mathbf{M}_{11}$ the new Hadamard matrix has order 88. Its portrait is shown on Fig. 2.

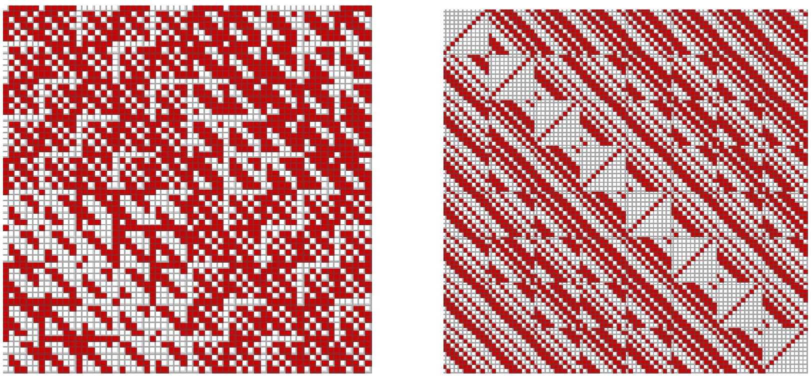

Fig. 2. Portraits of Hadamard matrixes of orders 60 and 88

The limitation of use of matrixes of orders $n=4t-1$ in the direct Scarpis method is due to the fact that Seidel matrix [5] is a core of order $n=4t-3$ of Belevitch matrix. The latter is the beginning of an independent chain of squarable Hadamard matrixes. It can't be expressed though matrices of lower orders. The inversed Scarpis method has no limitations for use of blocks **A** and **B** of orders $n=4t-1$ or $n=4t-3$. Block **B** of a composite matrix **M** can be either a Mersenne matrix or a Seidel matrix.

6 Conclusion

The generalization of the Scarpis method as a Kronecker product with correction, presented in this paper, broadens the general idea of the method. Initially it could be used only to multiply cores of Hadamard matrixes of prime orders by their natural extensions. In the presented modifications of the method the complexity of correction depends both on the type of symmetry (asymmetry), and on the place of factor matrixes in the product formula. Also, the product result can be obtained not only for factor matrixes of odd prime orders. The presented inversed Scarpis method that operates a removable border of normalized matrices, significantly broadens the possibilities of calculation of matrices of composite, higher orders.

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