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Quasi-Orthogonal

Local Maximum Determinant Matrices

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Abstract

Purpose: This note discusses quasi-orthogonal matrices which were first highlighted by J. J. Sylvester and later by V. Belevitch, who showed that three level matrices mapped to lossless telephone connections. The goal of this note is to develop a theory of such matrices based on preliminary research results. **Methods:** Extreme solutions (using the determinant) have been established by minimization of the maximum of the absolute values of the elements of the matrices followed by their subsequent classification. **Results:** We give the definitions of *Balonin-Sergeev* (BS) and *Cretan matrices* (CM), illustrations for some elementary and some interesting cases, and reveal some new properties. We restrict our attention in this remark to the properties of Cretan matrices depending on their order. This paper gives the set of basic definitions used in all our previous works, dedicated to Cretan matrices (quasi-orthogonal matrices with fixed levels). The suboptimal by determinant matrices has to have linear function, to describe numbers of their levels. We give the following estimation of this number as $\frac{n+1}{2} \pm m$, m≤1, levels. **Practical relevance:** Web addresses are given for other illustrations and other matrices with similar properties. Algorithms to construct Cretan matrices have been implemented in developing software of the research program-complex.

Mathematics Subject Classification: 05B20; 20B20

Keywords: Hadamard Matrices, Conference Matrices, Weighing Matrices, Constructions, Cretan Matrices

1 Introduction

We give a basic statements of theory, appeared due our works [1, 2, 3] dedicated to quasi-orthogonal local maximum determinant matrices.

The celebrated French mathematician Jacques Hadamard considered the following question (1893, [4]): among real matrices of order n with entries from the interval [–1, 1], find the matrices with maximum absolute value of the determinant (maximum determinant matrices, in short).

Maximum determinant matrices with entries ±1 and pairwise orthogonal columns (rows) were called *Hadamard matrices*, later due fundamental work of Paley [5] popularized this direction of matrix theory.

Hadamard matrices of order 2^m, m an integer, (not under this name) were known by Sylvester in at least 1857. Hadamard himself constructed examples of matrices of order of 12 and 20. He also conjectured that there exist such matrices of any order $n = 4t$, t an integer.

This conjecture of Hadamard is still open.

Determinant is function of all matrix entries, besides of points of global maximum, we can consider also points of local (relative) maximum of absolute value of the determinant of the same real matrices of order n with entries from the interval [–1, 1]. We call them *local maximum determinant matrices*. Local maximum determinant matrices with pairwise orthogonal rows have no this bound on entries (to be equal ±1), they exist for all orders n, n an integer. We call them quasi-orthogonal *local maximum determinant* matrices.

2 Hadamard matrix parameters

Hadamard matrices can be constructed with blocks having parameters, known as parameters of *difference sets* or *symmetric balanced incomplete block design* (SBIBD). We start this section with some definitions.

Definition 1. A (ν, k, λ) difference set in a multiplicative group G of order ν is k-subset D of G such, that every element $g \neq 1$ of G has exactly λ representations $g = d_1 d_2^{-1}$ with d_1, d_2 from G.

SBIBD operates the same set of parameters as characteristics of a block scheme. Hadamard matrix theory uses both these directions to construct binary matrix blocks, where ν – order of matrix or a matrix block, k – number of positive entries in every column (row), λ – number of positive entries with the same index in every two columns (rows). There is a version of *complementary parameters*, calculated for non positive entries, i.e. parameters of equivalent Hadamard matrix blocks. A traditional description uses $\nu > 2k$.

Definition 2. Paley parameters are (ν, *k*, λ) = (4*t*–1, 2*t*–1, *t*–1), *t* an integer.

They are known also as parameters of Hadamard designs. It is a main set for Paley's construction [5].

Definition 3. Menon parameters are (ν, *k*, λ) = ($4u^2$, $2u^2–u$, $u^2–u$), *u* an integer.

It is parameters of *regular Hadamard matrices*, i.e. matrices with the same sums of columns (rows), or parameters of Menon difference sets [6]. These difference sets are called Hadamard difference sets (HDS) [7], since their (1, − 1)-incidence matrices are Hadamard matrices.

Definition 4. The parameters of a PG (*q*, *m*) projective geometry or Singer difference sets are $(v,k,\lambda)=(\frac{q^{m+1}-1}{q-1},\frac{q^{m}-1}{q-1},\frac{q^{m-1}-1}{q-1})$, *q* a prime power.

Parameters of difference sets, appeared first (in additive notations) due to James Singer (1938, [8]), were illustrated later by an extensive survey of Marshall Hall in 1956 [9].

There are also McFerland parameters, Spence parameters, Chen-Davis-Jedwan parameters, ets.

In all these cases Hadamard matrices are described by parametric functions connected with size of these binary matrices or size of their blocks. Binary (or trinary) quasi-orthogonal *local maximum determinant matrices* have values of entries *a*=1, –*b*, and so on; these parameters are different with Hadamard ones and have some own sets of functions.

3 Quasi-orthogonal matrices

Definition 1. The values of the entries of a matrix are called levels.

Hadamard matrices [10] are two-level matrices, symmetric conference matrices [11] and weighing matrices [12] are three-level matrices, for example.

Definition 2. An *Hadamard matrix* of order *n* is a square matrix $\mathbf{H}_n$ with elements {1, –1} such, that $\mathbf{H}_n^T\mathbf{H}_n = n\,\mathbf{I}$, where **I** is the identity matrix.

Hadamard matrices can only exist for orders 1, 2 and *n*=4*t*, *t* an integer (the so called *Hadamard conjecture*). The Hadamard inequality [4] says, that Hadamard matrices have maximal determinant for the class of matrices with entries from the unit disk (interval [–1, 1]) – the moduli of the elements are less or equal 1 by default.

Definition 3. A *symmetric conference matrix* of order *n* is a square matrix **C** with elements 0, +1 or −1, satisfying $\mathbf{C}_n^{\mathbf{T}}\mathbf{C}_n = (n-1)\,\mathbf{I}$ **.**

Symmetric conference matrices [11] can only exist for orders *n*=4*t*−2, *t* an integer, if the number *n*–1 is the sum of two squares. Similar to symmetric conference matrices are square matrices **W**, with elements {0, ±1}, satisfying

$\mathbf{W}_n^{\mathbf{T}}\mathbf{W}_n = (n - m)\ \mathbf{I}$. These are called *weighing matrices* $\mathbf{W}(n, n - m)$ [12]. The class of *quasi-orthogonal* matrices with maximal determinant and entries from the unit disk may have a very large set of solutions [2].

Definition 4. A real square matrix $\mathbf{X}_n$ of order n is called *quasi-orthogonal* if it satisfies $\mathbf{X}_n^{\mathrm{T}}\mathbf{X}_n = \omega\ \mathbf{I}$, where $\omega \le n$ is a constant real number.

In this work we will only quasi-orthogonal matrices with real elements and a least one entry in each row and column must be 1 [1]. Hadamard matrices are the best known of these matrices with entries from the unit disk. Symmetric conference matrices, a particularly important class of quasi-orthogonal matrices.

Quasi-orthogonal matrices with maximal determinant of odd orders have been discovered to have a larger number of levels. There are only five matrices $\mathbf{A}_3$, $\mathbf{A}_5$, $\mathbf{A}_7$, $\mathbf{A}_9$, $\mathbf{A}_{11}$ with $\frac{n+1}{2} \pm m$, $m \le 1$, moduli of levels [2].

4 Quasi-orthogonal matrix conjectures

The main conclusion, that can be made due previous extensive researches [1, 2, 3], says: number of quasi-orthogonal *maximum determinant matrices* with reasonable number of levels are bounded by the first five examples.

Conjecture 1. There are only five quasi-orthogonal *maximum determinant matrices* matrices $\mathbf{A}_3$, $\mathbf{A}_5$, $\mathbf{A}_7$, $\mathbf{A}_9$, $\mathbf{A}_{11}$ with $\frac{n+1}{2} \pm m$, $m \le 1$, moduli of levels, shown on Fig. 1.

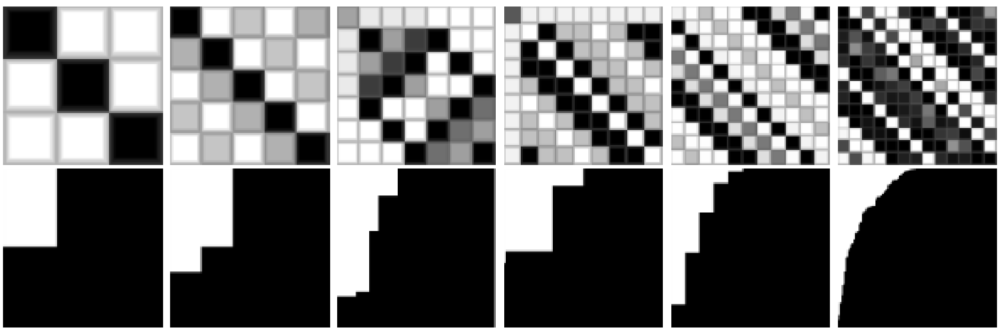

Fig. 1. Portraits and histograms of quasi-orthogonal matrices $\mathbf{A}_3$, $\mathbf{A}_5$, $\mathbf{A}_7$, $\mathbf{A}_9$, $\mathbf{A}_{11}$, $\mathbf{A}_{13}$

Let a=1, b, c, … be moduli of levels. Then A3 is circ(–b, a, a), i.e. circulant matrix constructed by shown sequence. A5 is circ(–a, a, b, a, c). The next complicated matrix A7 has the first row and column [d, b, b, b, a, a, a] and four-blocks core [[backcirc(–a, e, –c), circ(–a, a, a)],[circ(–a, a, a), backcirc(e, –a, –d)]], and matrix A9 has the first row and column [–d, b, b, b, b, b, b, b, b]; core circ(a,–a, c, c, a, c,–a,–a). Matrix A11 returns to circ(c, a, e, a, a, –a, –a, d, –f, b, –

a). And, due conjecture, it is the last matrix with simple structure. Matrix A13 goes off a common rule.

Conjecture 2. Quasi-orthogonal matrices with near to maximum absolute maximum of determinant have $\frac{n+1}{2}$ moduli of levels.

This question need to more detail research, but it is proven: optimal matrices with $\frac{n+1}{2}\pm 1$ moduli of levels have suboptimal twins with $\frac{n+1}{2}$ moduli of levels for orders 8, 9; besides, there are such versions for orders 13 and more.

5 Cretan matrices

It was shown, that optimal and suboptimal quasi-orthogonal matrices has too much number of levels, that exceed with some reasonable bounds to be easy found and classified by types, as it was made for Hadamard matrices.

Definition 1. A quasi-orthogonal matrix with extremal or fixed properties: global or local extremum of the determinant, saddle points, the minimum number of levels, or matrices with fixed numbers of levels is called a *Balonin-Sergeev* matrix (*BS-matrix*) [1]. They are researched in [2, 3, 14 – 18].

It is a wide class of possible matrices. Balonin-Sergeev matrices with fixed (for any order) numbers of levels were first mentioned during a conference in Crete, they are called *Cretan matrices* (CM-*matrices*) [1].

Definition 2. A *Cretan matrix* $\mathbf{X}_n$, of order n, is quasi-orthogonal matrix with fixed number of levels, it satisfies $\mathbf{X}_n^{\mathrm{T}}\mathbf{X}_n = \omega(n)\ \mathbf{I}$, where $\omega(n) \le n$ is the weight, and defined by a function $\omega(n)$ or functions $a(n)=1$, $b(n)$, $c(n)$, … of its levels. We write CM (n; $b(n)$, $c(n)$, ..; $\omega(n)$) as shorthand.

Determinant of *Cretan matrix*, $\det(\mathbf{X}) = \omega(n)^{n/2}$.

Normalized determinant $\psi(n) = [\omega(n)/\mathrm{n}]^{1/2}$; $\omega(n)=n\psi(n)^2$, so $\psi(n)=1$ for all Hadamard matrices.

Hadamard inequality gives an estimation of determinant $\det(\mathbf{X}_n)\le n^{n/2}$, while $\det(\mathbf{X}_n)=n^{n/2}\times\psi(n)^{\,n}$, gives a possible value.

Some papers operate also so called *Hadamard norm of Cretan matrix* $h(n)=\psi(n)^{-1}$ [2, 3]. Hadamard matrices have minimal possible value $h(n)=1$, all other quasi-orthogonal matrices have more big norm.

Any quasi-orthogonal matrix can be made orthogonal through product $h(n)\mathbf{X}_n$, so $h(n)$ describes the maximum of absolute value of entries of orthogonal matrix.

The task to find a matrix with minimum of its maximal element gives a minimax matrix, so quasi-orthogonal matrices with absolute (or relative) maximum of moduli of determinant can be observed as minimax matrices, and this definition open the way to construct effective numerical methods to search extremal structures Cretan matrices including.

6 Cretan matrix families

Balonin and Sergeev concluded [2, 3] that the resolution of the question of the existence of quasi-orthogonal matrices depends on the order:

- for $n = 4t$, t an integer, at least 2 levels, a, $-b$, $|a| = |b|$, are needed;
- for $n = 4t-1$, at least 2 levels, a=1, –b, $b < a$, are needed;
- for $n = 4t-2$, at least 2 levels, a=1, –b, $b < a$, are needed for a two block construction;
- for $n = 4t - 3$, at least 3 levels, a=1, $-b$, d, $b < a$, $d < a$, are needed.

We give the following list of Cretan matrix families.

Hadamard matrix. A Cretan matrix, named Hadamard matrix [1], **H**, is a two-level quasi-orthogonal matrix of order n with level functions a=1,–b; b=1.

They noted CM (4t;1) or CM(4t;1;ω), Second notation for these matrices is handy for a concrete matrix order.

Hadamard conjecture: Hadamard matrices exist for n=1,2 and 4t, t an integer [4]. Orders of *elementary Hadamard matrices* cover $n = 2^m$, m an integer.

Mersenne matrix. A Cretan matrix, named Mersenne matrix [2, 13, 14], **M**, is a two-level quasi-orthogonal matrix of order n with level functions a=1,–b; where $b = \frac{t}{t+\sqrt{t}}$, $t=(n+1)/4$.

They noted CM (4t–1; $\frac{t}{t+\sqrt{t}}$).

Balonin conjecture [13]: Mersenne matrices exist for $n=4t-1$, t is integer.

For so called *elementary Mersenne matrices* set (or *pure* Mersenne matrices) orders cover Mersenne numbers $n = 2^m - 1$, m an integer.

Invariant of Mersenne matrices is difference 1 between numbers of positive and negative entries in every column (row), so weight of these matrices $\omega=(n+1)/2+(n-1)b^2/2=2t+(2t-1)b^2$.

Euler matrix. A Cretan matrix, named Euler matrix [2, 13, 15], **E**, is a four-level quasi-orthogonal matrix of order n, it can be observed as two-circulant or two blocks A, B matrices with two levels a=1, –b; where $b = \frac{t}{t+\sqrt{2t}}$, $t=(n+2)/4$.

They noted CM(4t–2; $\frac{t}{t+\sqrt{2t}}$).

Balonin conjecture [13, 14]: Euler matrices exist for $n=4t-2$, t an integer.

Their orders cover singular orders for the symmetric conference matrices: the latest matrices do not exist if $n-1$ is not sum of two squares.

Invariant of Euler matrices is difference 2 between numbers of ±1 and ±b entries in every column (row), so weight $\omega = (n+2)/2+(n-2)b^2/2=2t+2t-1)b^2$.

Seidel matrix. A Cretan matrix, named Seidel matrix [2], S, is a quasi-orthogonal matrix of order n with level functions a=1, –b, d; where second level b=1–2d, and third level $d = \frac{1}{1+\sqrt{4t-3}}$ belongs to diagonal entries, t=(n+3)/4.

They noted CM (4t–3; 1–2d, $\frac{1}{1+\sqrt{4t-3}}$).

Their orders cover singular points of symmetric conference matrices; Seidel matrices do not exist if n is not sum of two squares [8].

Invariant of Seidel matrices is difference 1 between numbers of positive and negative entries in every column (row), so weight of these three level matrices is $\omega=(n-1)(1+b^2)/2+d^2 = 2(t-1)(1+b^2)+d^2$.

Fermat matrix. A Cretan matrix, named Fermat matrix [2, 16], F, is a quasi-orthogonal matrix of order n with level functions a=1, –b, s; for n=3, 2a=b=s=1; for n>3, $q=n-1=4u^2$, $p = q+\sqrt{q}$, we have $b \le s < a$; where the second matrix level $b = \frac{2n-p}{p} = \frac{2u^2-u+1}{2u^2+u} = 1-\frac{2u-1}{2u+1}\times\frac{1}{u}$, and level $s = \frac{\sqrt{nq-2\sqrt{q}}}{p} = \frac{\sqrt{nu-1}}{2u+1}\times\frac{1}{\sqrt{u}}$ is for border entries.

They noted CM ($4u^2$+1; $1-\frac{2u-1}{2u+1}\times\frac{1}{u}$, $s = \frac{\sqrt{nq-2\sqrt{q}}}{p} = \frac{\sqrt{nu-1}}{2u+1}\times\frac{1}{\sqrt{u}}$).

For so called *elementary Fermat matrices* set (*pure* Fermat matrices) orders cover Fermat numbers $n = 2^{2^q}+1$, q an integer.

Invariant of Fermat matrices: core has $k = \frac{q-\sqrt{q}}{2} = 2u^2-u$ entries a=1, weight $\omega(n) = k+(q-k)b^2+s^2 = 1+4u^2s^2$.

7 Conclusion

This paper gives the set of basic definitions for Cretan matrices (quasi-orthogonal matrices with fixed levels). The suboptimal by determinant matrices has to have $\frac{n+1}{2}$ moduli of levels. These matrices are closely associated with Hadamard and weighing matrices. The questions of existence of Cretan matrices observed by works [2, 3]. There is a popular Hadamard-type matrices catalogue created with active participations of authors of paper [19].

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References

[1] Balonin N. A., and Seberry, Jennifer, Remarks on extremal and maximum determinant matrices with real entries ≤ 1. *Informatsionno-upravliaiushchie sistemy,* № 5(71) (2014), pp. 2–4 (In English).

[2] Balonin N. A., Sergeev M. B., Local Maximum Determinant Matrices. *Informatsionno-upravliaiushchie sistemy*, 2014, № 1 (68), pp. 2–15 (In Russian).

[3] Balonin N. A., Sergeev M. B., On the Issue of Existence of Hadamard and Mersenne Matrices. *Informatsionno-upravliaiushchie sistemy*, 2013, № 5 (66), pp. 2–8 (In Russian).

[4] Hadamard J., Résolution d'une question relative aux déterminants. Bulletin des Sciences Mathématiques, 1893. Vol. 17, pp. 240–246.

[5] Paley R. E. A. C., On orthogonal matrices. *J. of Mathematics and Physics*, 1933, Vol. 12, pp. 311–320.

[6] Menon, Kesava, P., On difference sets whose parameters satisfy a certain relation," Proc. Amer. Math. Soc. vol. 13, 1962, pp. 739–745. http://dx.doi.org/10.1090/s0002-9939-1962-0142471-0

[7] Jonathan Jedwab, James Davis. A survey of Hadamard difference sets. Hewlett Packard Technical Reports; University of Richmond, Richmond, VA23173, 1994, pp. 1–16.

[8] Singer, J. A theorem in finite projective geometry and some applications to number theory, Trans. Amer. Math. Soc. 43 (1938) 377–385. http://dx.doi.org/10.1090/s0002-9947-1938-1501951-4

[9] M. Hall, Jr., A survey of difference sets, Proc. Amer. Math. Soc. 7, 1956. pp. 975–986. http://dx.doi.org/10.1090/s0002-9939-1956-0082502-7

[10] Seberry, Jennifer, Yamada, Mieko. Hadamard matrices, sequences, and block designs, *Contemporary Design Theory*: *A Collection of Surveys*, J. H. Dinitz and D. R. Stinson, eds., John Wiley and Sons, Inc., 1992. pp. 431–560.

[11] Balonin N. A. and Jennifer Seberry, A review and new symmetric conference matrices. *Informatsionno-upravliaiushchie sistemy*, 2014, № 4(71), pp. 2-7 (In English).

[12] Wallis (Seberry), Jennifer. Orthogonal (0, 1, –1) matrices. *Proceedings of First Australian Conference on Combinatorial Mathematics*, TUNRA, Newcastle, 1972, pp. 61–84.

[13] Balonin N. A. Existence of Mersenne Matrices of 11th and 19th Orders. *Informatsionno-upravliaiushchie sistemy*, 2013, № 2, pp. 89 – 90 (In Russian).

[14] Sergeev A.M. Generalized Mersenne Matrices and Balonin's Conjecture. *Automatic Control and Computer Sciences*, 2014. Vol. 48, № 4. pp. 214–220. http://dx.doi.org/10.3103/s0146411614040063

[15] Balonin N. A., Sergeev M. B. Two Ways to Construct Hadamard-Euler Matrices. *Informatsionno-upravliaiushchie sistemy*, 2013, № 1(62), pp. 7–10 (In Russian).

[16] Balonin N. A., Sergeev M. B. Mironovsky L. A. Calculation of Hadamard-Fermat Matrices. *Informatsionno-upravliaiushchie sistemy*, 2012, № 6(61), pp. 90–93 (In Russian).

[17] Balonin N. A., Sergeev M. B. Matrix of Golden Ratio G10. *Informatsionno-upravliaiushchie sistemy*, 2013, № 6 (67), pp. 2–5 (In Russian).

[18] Balonin Yu. N., Sergeev M. B. M-matrix of 22nd order. *Informatsionno-upravliaiushchie sistemy*, 2011, № 5(54), pp. 87–90 (In Russian).

[19] Balonin N. A., Djokovic D. Z., Mironovski L.A., Seberry Jennifer, Sergeev M. B. Hadamard type Matrices Catalogue, available at: http://mathscinet.ru/catalogue (accessed 5 november 2014).

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