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Quasi-Orthogonal Matrices with Level Based on Ratio of Fibonacci Numbers

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Abstract

This paper discusses quasi-orthogonal matrices which were first highlighted by J. J. Sylvester and J. Hadamard, who showed that two level matrices exist for even orders 4*t*, *t* integer. We consider two-level matrices complementing the Hadamard, Mersenne and Euler matrices. We give definitions of golden ratio matrices as illustrations for some elementary and interesting cases, and reveal some new properties. The definitions of a section and a layer of quasi-orthogonal matrices are provided. The example of continuous matrices with varying levels is used to show, that the golden section matrices branch is closely associated with Hadamard and conference matrices. Commentaries on the applied aspects of the golden section matrices use are provided here as well.

Mathematics Subject Classification: 05B20; 20B20

Keywords: Hadamard matrices, conference matrices, Mersenne matrices, Euler matrices, Fibonacci numbers, golden ratio, golden ratio matrices

1 Introduction

The Fibonacci rule F(*n*)= F(*n*–1) + F(*n*–2) with initial conditions F(0)=F(1)=1 allows to generate the Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, … . Among "Fibonacci-like” sequences there are Lucas numbers which can be calculated with F(0)=2, F(1)=1. The resulting sequence 2, 1, 3, 4, 7, 11, 18,… looks quite different from the previous one, but the ratio of numbers converges toward the golden ratio, just as ratio of Fibonacci numbers themselves do. Any Fibonacci-like sequence can be expressed as a linear combination of both sequences. There are starting pairs for which we can get a ratio different from the golden ratio x, $x^2 = x + 1$, but they are rare. The quadratic equation $x^2 - x - 1 = 0$ has two roots

$$x_1 = \frac{1+\sqrt{5}}{2} = 1.618\ldots \text{ and } x_2 = \frac{1-\sqrt{5}}{2} = -0.618\ldots \quad ,$$

the first is recognized as a golden ratio and from $x^2 - (x_1+ x_2)\, x + x_1x_2 = 0$ we have $x_1x_2 = -1$, so these two solutions are inversed by sign and value $x_2 = -1/ x_1$.

Works [1–3] observe orthogonal matrices with orders or tiers equal to some specific numbers of number theory.

A *quasi-orthogonal matrix*, order n, is a square matrix **A**, $|a_{ij}|\leq 1$, with maximum modulus 1 in each column (and row). It fulfills $\mathbf{A}^T\mathbf{A} = \omega(n)\mathbf{I}$, with **I** the identity matrix and $\omega(n)$ the *weight*.

The entries values will be called matrix “tiers” or “levels”. An Hadamard matrix with entries {1, –1} is a two-level matrix. A Mersenne matrix with entries {1, $-b$}, $0<b<1$ is also a two-level matrix.

The Mersenne matrices are two-level quasi-orthogonal matrices defined by their second level $-b = -\frac{p}{p+\sqrt{p}}$, $p = \frac{n+1}{4}$, n – order of matrix. Authors shown [1, 2], that there is a simple rule to generate such matrices for all orders n = 3, 7, 11, … that is: the *Mersenne numbers* $m = 2^t-1$, t integer.

Here $p = \frac{n+1}{4}$ is a fundamental number which plays a big role in the Hadamard matrix theory. The numerator $n + 1$ is corresponding Hadamard matrix order: every Mersenne matrix, with entries {1, $-b$}, is a *core* of an Hadamard matrix, with entries {1, –1}, i.e. negative level is changed to smaller value.

While Hadamard matrix and Hadamard-type matrices are non orthogonal matrices, in the strong sense that they $\mathbf{A}^T\mathbf{A} = \mathbf{I}$, we will name them orthogonal in short.

There are also ortogonal *Fermat* matrices, which orders are defined by *Fermat numbers*, and ortogonal *Euler* matrices, connected with a replacement of Hadamard-type matrices (conference matrices) when its order does not satisfy the generalized Fermat-Euler criteria of existence, connected with the well known sum of two squares rule [4].

Now we are interested in orthogonal matrices, which entries equal to the golden ratio inverse value $x_2 = -1/ x_1 = -0.618$ ….

We will consider such orthogonal matrices with orders 5, 10 and proportional to 10 values: 10, 20, 40, 80, 160, 320, 640,.. usual in image processing algorithms, since they are popular photo formats [5]. Besides, number 10 is a foundation of geometrical figures (pentagrams) and it is connected with golden ratio. Such theoretical approach of matrix and number theory tie was developed in our works [1, 3].

The quadratic equation $x^2 + x - 1 = 0$ has inversed roots: if we discuss level modulus less then 1, we will take as a main solution $g = 0.618 \ldots < 1$.

2 Definitions

The mentioned theoretical approach and analysis of the conditions of existence of *Mersenne* matrices [1, 3] has raised the question of how to draw (in an economical way) all quasi-orthogonal matrix family, specially the particular cases of *Hadamard*, *Mersenne*, *Euler* and *Fermat* matrices [1]. These matrices are listed in the descending order of the d-variable, d=0, 1, 2, 3, in the formula for their sizes $n=4k-d$.

Definition 1. In this paper a *matrix layer* is a set of quasi-orthogonal matrices, with a known function of the entries describing their dependence on $n = 4k - d$ for some d and all possible $k>0$.

A Mersenne matrix, of order n, has negative entries $-b$, described by some function of moduli $b = f(n)$ and determined for all orders $n = 4k - 1$. Any Mersenne matrix belongs to this layer. In the same way, Hadamard and Euler matrices with sizes $n = 4k - d$, d=0, 2, as described in [1], belong to some layers.

Fermat matrices do not constitute such a layer, as their level function is defined within a narrow set of values $n = 2^k + 1$ for even and some odd values of integer k.

***Definition* 2.** In this paper a *section* is a set of quasi-orthogonal matrices of different layers, which depend on $n = 4k - d$ for some k and all possible d=0, 1, 2, 3.

The matrices mentioned above are the manifestation of a mathematical object, described by its layers and sections. The existence of any matrix in a section requires the existence of all other matrices of the same section because these matrices are mutually dependent.

Besides Hadamard matrices with entries {1, –1} and similar to them Mersenne matrices with entries $\{1, -b\}$, $0<b<1$, there are other matrices with small numbers of levels. The Euler matrix is a quasi-orthogonal matrix with entries $\{\pm 1, \pm b\}$, defined by modulus of second level $b = \frac{p}{p+\sqrt{2p}}$, $p = \frac{n+2}{4}$, where n – order of matrix..

The number of levels (tiers) is an important characteristic of a matrix set.

For example, too low number of levels does not guarantee existence of three-level conference matrices (Belevitch matrices) [4]. They do not exist for order $m=4p-2$, if $m-1$ is not the sum of two squares.

The number of matrix levels (tiers) increases with the value d in $n=4k-d$. Hadamard matrices have single level (by modulus of elements) matrices as the elements are 1 or –1 [6, 7]. Mersenne matrices are two-level matrices; Euler matrices are four–level matrices. All these matrices have some minimal number of levels guaranteeing their existence for pre chosen orders [8].

Many sets of quasi-orthogonal matrices with low numbers of levels do not belong to a layer. They are special orthogonal per columns (Hadamard type) matrices: conference matrices with three levels of entries {0, 1, –1} are defined for orders shared with the bigger family of four levels Euler matrices. Paley [7] noted that any Hadamard matrix (or quasi-orthogonal matrix respectively) can be used to give the same type matrix of the double size using the Sylvester algorithm. We name them Sylvester constructions.

In this case, a new matrix branch appears: it does not intersect with any of the previous branches. The Paley's observation induces us to study *artifact matrices* from the orthogonal matrix family (the Hadamard family), including the *golden ratio matrices.* These are considered in this paper.

3 Continuous Matrices

Continuous matrices are different from the orthogonal (Hadamard) family seen in former section. Their level functions depend on more than one argument n. Therefore, they generate not one, but a continuum of quasi-orthogonal matrices, described by a parametric dependence.

This possibility follows from the interpretation of orthogonal or quasi-orthogonal matrix as a table of vector projections of the required orthogonal basis. We use optimal to denote matrices with maximal determinant. This allows us to get non-varying matrices for this continuum, known as orthogonal (Hadamard) matrices [6].

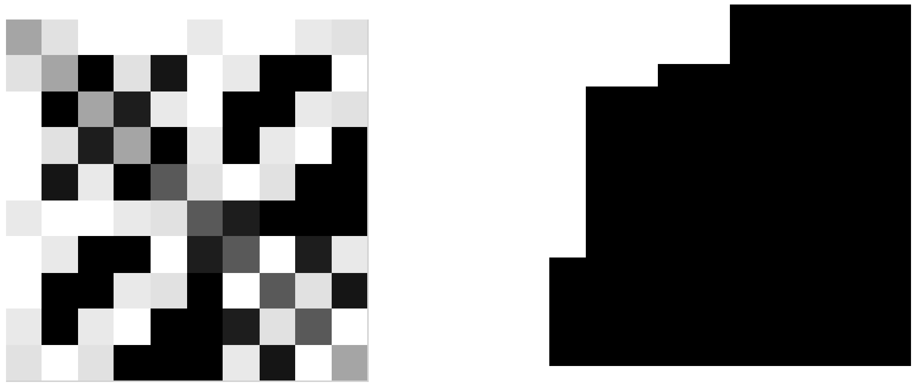
Sub-optimal solutions are known as quasi-orthogonal matrices [2, 3] with a small number of levels. Fig. 1 shows a continuous matrix $\mathbf{M}_{10}$. The brightness of a cell represents the value of the level of its element between zero (white) and 1 (black).

Let's denote levels of quasi-orthogonal matrix as $a \geq b \geq c \geq g$.

We will rank matrices by levels. The upper level is a=1. The second and the third levels depend on the lower level g as $b^2+2(b-1)+2(g-c)+c^2=0$, $c=1/(g+1)$.

The continuous matrix $\mathbf{M}_{10}$ is a matrix with a low number of changeable levels and it is remarkable by special solutions: two bounds of a continuum.

One solution is the conference matrix [4] $\mathbf{C}_{10}$. When $b=c=a=1$ we have diagonal entry g=0.



a) *b)*

Fig. 1. Portrait of matrix $\mathbf{M}_{10}$ (*a*) and histogram of moduli of its elements (*b*)

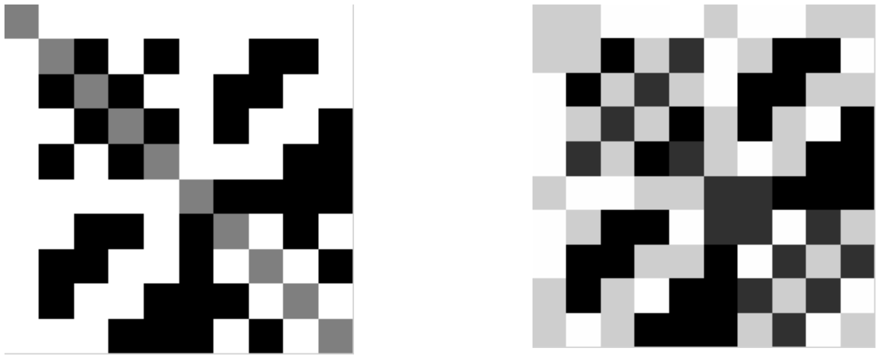
We name the second bound of this continuum as *a gold ratio matrix* $\mathbf{G}_{10}$, because when $b=c=g<a=1$:

$$\mathbf{G}_{10} = \begin{pmatrix} g & g & a & a & a & g & a & a & g & g \\ g & g & -a & g & -g & a & g & -a & -a & a \\ a & -a & g & -g & g & a & -a & -a & g & g \\ a & g & -g & g & -a & g & -a & g & a & -a \\ a & -g & g & -a & -g & g & a & g & -a & -a \\ g & a & a & g & g & -g & -g & -a & -a & -a \\ a & g & -a & -a & a & -g & -g & a & -g & g \\ a & -a & -a & g & g & -a & a & -g & g & -g \\ g & -a & g & a & -a & -a & -g & g & -g & a \\ g & a & g & -a & -a & -a & g & -g & a & g \end{pmatrix}.$$

It is distinguished by the equation $g^2+g-1=0$ followed from the condition of orthogonality and well known by its irrational roots. Number called *golden ratio* in the Fibonacci numbers theory 1.618... exceed 1, so in this case we are interested in the lower level g=0.618.. (so we use here the equation for this positive solution).

Matrix portraits give presentations of the conference matrix $\mathbf{C}_{10}$ and golden ratio matrix $\mathbf{G}_{10}$, see Fig. 2. There is a two-circulant form of both these matrices: we publish now the new matrix form connected with oscillation of entry values in matrix columns.

Such kind of oscillation is used to generate well known Walsh-functions inside the Hadamard matrix structure [3].



a) *b)*

Fig. 2. Portraits of the conference matrix $\mathbf{C}_{10}$ (a) and golden ratio matrix $\mathbf{G}_{10}$ (b)

4 The Family of golden ratio matrices

So, all golden ratio matrices are defined on orders n=10·2^k. For them, as for all Hadamard family matrices, matrix $\mathbf{G}_{10}$ is the starting point for the sequence of matrices, found by iterations

$$\mathbf{G}_{2n} = \begin{pmatrix} \mathbf{G}_n & \mathbf{G}_n \\ \mathbf{G}_n & -\mathbf{G}_n \end{pmatrix}.$$

The value of modulus level g is constant. This implies that the golden ratio matrices and Hadamard matrices are two boundary solutions of a continuum matrix, see Fig. 3.

a) *b)*

Fig. 3. Portraits of Hadamard matrix $\mathbf{H}_{20}$ (*a*) and golden ration matrix $\mathbf{G}_{20}$ (*b*)

The golden ratio matrices coexisted with conference (Belevitch) and Euler matrices by their definition. They are quasi-orthogonal matrices of even orders.

Histograms of the element modulus of the Euler matrix $\mathbf{E}_{10}$ (Fig. 3, *a*) and matrix $\mathbf{G}_{10}$ (Fig. 3, *b*) suggest that these two-level by modulus of entries matrices are similar. However, these matrices are significantly different. The golden ration matrix level is a constant, by definition, while the Euler matrix level is a function of its order. So the first matrix can be calculated faster.

a) *b)*

Fig. 3. Histograms for Euler matrix $\mathbf{E}_{10}$ (a) and golden ratio matrix $\mathbf{G}_{10}$ (*б*)

5 Conclusion

This paper describes a golden ratio matrix $\mathbf{G}_{10}$ and a sequence of such G-matrices, represented by a shown example: $\mathbf{G}_{20}$. These matrices are closely associated with Hadamard and conference matrices, their specific structures and the algorithms to find them.

The range of application of mathematical models as orthogonal bases is wide. There is a curious idea to use the continuous matrix as a model of phase transformations taking place during the crystallization of cooled alloys. Two level golden ratio matrices can be a model reflecting the details of crystal structures. Golden ratio matrices, represented by the starting matrix $\mathbf{G}_{10}$, connect with Hadamard and conference matrices as bounds of continuum. The golden ratio matrices have no a layer by the definition, but they have orders $n=10\cdot 2^k$. These sizes 10, 20, 40, 80, 160, 320, 640, etc. hold a special place in image processing algorithms.

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