Conference Matrices with Two Borders and Four Circulants

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Dedicated to the Memory of Scott Vanstone

Abstract

A unified introduction to rich and poor structures of conference matrices is give using a system of invariants of two border and four circulant matrices. Conference matrix portraits with circulated entries are presented to emphasize the visulaizations of constructions. The difference between structures with odd and even orders of cells is highlighted via illustrations of examples of rich A frequency separation of cells is noted.

Key Words and Phrases: Conference matrices, Hadamard matrices, circulant difference sets, symmetric difference sets, constructions, telephony.

AMS Subject Classification: 05B20; 20B20.

1 Introduction

Consider a two border multicirculant conference matrix [1] of order $4t + 2$

$$\mathrm{C}=\left[\begin{array}{cc|cc} 0 & 1 & e & e \\ 1 & 0 & e & -e \\ \hline e^\top & e^\top & S & G \\ e^\top & -e^\top & G^\top & -S \end{array}\right]$$

where

$$\mathrm{S}=\begin{bmatrix} A & B \\ B^\top & A \end{bmatrix} \qquad \text{and} \qquad \mathrm{G}=\begin{bmatrix} C & D \\ F & E \end{bmatrix}$$

with A, B, C, D, E, F circulant or back-circulant cells of size $t \times t$, $A^\top = A$, e is the $1 \times t$ matrix of all ones.

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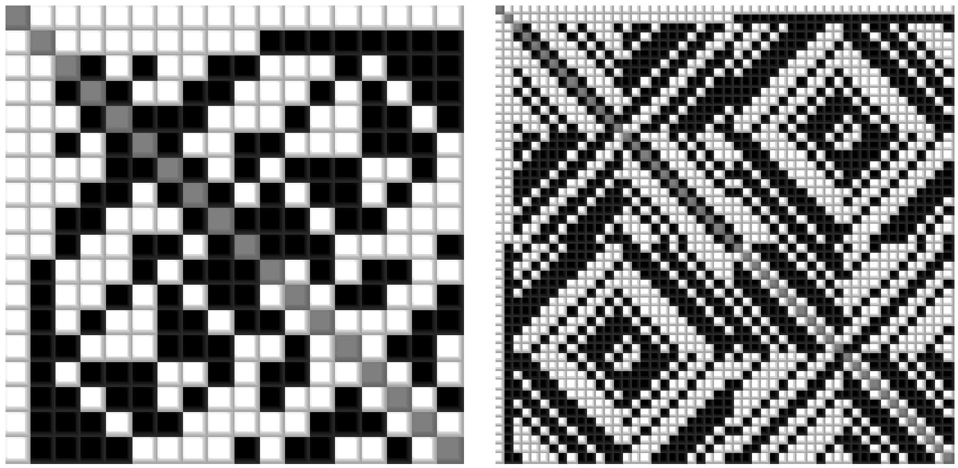

Figure 1: Matrices C_{18} and C_{50} with circulated entries

We will call it *two border four circulant conference matrix* if the two last cells E, and F, depend on the previous ones C, D. The simplest construction consists of E=C, F=D (or circulant shifted C, D).

Let T be the usual shift matrix $T = (t_{ij})$, $t_{ij} = 1$, if $j - i = 1 \pmod t$, $t_{ij} = 0$ otherwise, and R be the usual back-shift matrix (reversed diagonal matrix): $R = (r_{ij})$, $r_{ij} = 1$, if $i + j = 1 \pmod t$, $t_{ij} = 0$ otherwise. So circulant shifted D is matrix $F = DT^x$, for some non-negative integer x.

2 Circulant Matrix with Circulated Entries

A conference matrix with both types of cells (circulant and back-circulant ones) we will call a matrix with *rich structure* [2]. The matrix will be said to have *poor structure* if it has circulant cells only.

The rich structure with circulant A, D, F and back-circulant B, C, E can be seen as a matrix with circulated entries (Fig. 1). It remembers the phase portraits of dynamical systems and reflects the structure of kernels of integral operators of Fourier-type.

This interpretation (flows of entries) is new and it can explain some structural (ornamental) invariants we will see further. We will call such a flow-structure *"curl of Seberry"*: it appeared first at works of Jennifer Seberry with skew-Hadamard matrices [7], [8].

The skew Hadamard matrices are given at:
http://mathscinet.ru/catalogue/skewhadamard

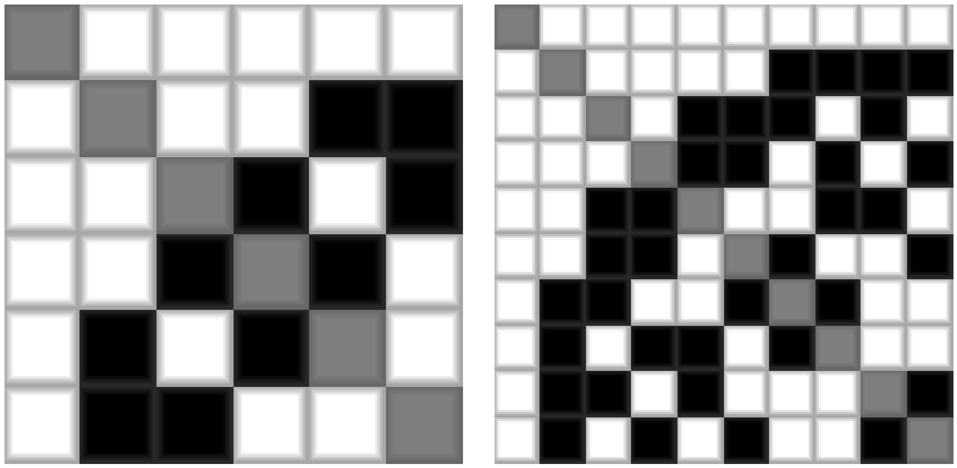

Figure 2: Initial matrices C_6 and C_{10}

3 The Trivial Orders

The trivial orders include matrices C_6 and C_{10}. The initial matrices have no specific features: they are not categorized by types using circulant or back-circulant cells (Fig. 2). Note, that C_{10} has a shifted block $F = DT$.

4 The Odd Line Family of Matrices

The simplest construction of matrix with circulated entries consists of a back-circulant $E = RCR$ and circulant $F = RDR$ cells (mirror images).

We say this line of orders, $4t + 2$, where t is odd, is the *odd line*: 6, 14, [22 does not exist], 30, 38, [46 is a special case], 54, ...

5 The Even Line of Family of Matrices

The simple construction described above, cannot be used for orders 10 or 18, but exists for order 26.

We say this line of orders, $4t + 2$, where t is even, is the *even line*: it is different from the previous one by having some shifted cells $E = RCRT^x$ or/and $F = RDRT^x$.

Among these matrices of noticible (by symmetry distruction) type we observe the one with orders: 10, 18, 26 is a special case (it has matrices with *ornamental invariants* of the odd line), [34 does not exist], 42, 50, [58 does not exist], ...

6 The System of Invariants

Conference matrices with one border and circulant core have a system of invariants using matrix symmetry [5]. The system of invariants allows us to make a fast computer search for such matrices [6].

Conference matrices with two borders also have a system of invariants. Let us note that $A^\top = A$.

This system will be more complete with the invariants of B.

The odd line of matrices is the simplest. Besides A, the matrix B can be symmetric: $B^\top = B$. The rich structure has the same property $RB^\top = BR$ (the symmetry using the reversed diagonal).

The even line of matrices of orders $4t+2$, t is even, has its matrix B based on symmetric or asymmetric sequences: so $\{\pm 1 \ b \pm 1 \pm bR\}$ or $\{b \pm 1 \pm bR \pm 1\}$ for the rich structure, where b is a (1, -1)-subsequence.

7 The Cross-Invariants

Equations for the pairs C, E and D, F are additional matrix invariants. We will call them *ornamental* because they describe the specifics of a matrix portrait.

Pairs C and D can be connected too. The simplest structure has a flip-inversed construction: $D = -CR$, t is even, it is illustrated for C_{18} (Fig. 3).

The back-circulant C and circulant D can be built on the pair of inversed sequences: c and $d = -c$, it is seen for C_{30} (Fig. 3).

8 Matrices with Rich Structure

Let us consider images of matrices of rich structure (Fig. 3). This structure is tightly connected with the poor one, it can be based on the same or circulant shifted sequences to calculate A, B, C, D matrices.

The row and column permutations of S and G (made to produce the rich structure) do not preserve the corresponding appearance of the block $-S$: but additional permutations in this area lead to the required results. These structures can have equivalent matrices.

The rich structure has some self coherance due the original ornamental content – it follows from the flows of entries.

These are given at:

http://mathscinet.ru/catalogue/conference/twobordersfourcirculant

9 Matrices with Poor Structure

We consider images of matrices of poor structure (Fig. 4). These are the results of a computer search that used the described system of ornamental invariants.

These are given at:

http://mathscinet.ru/catalogue/conference/twobordersfourcirculant

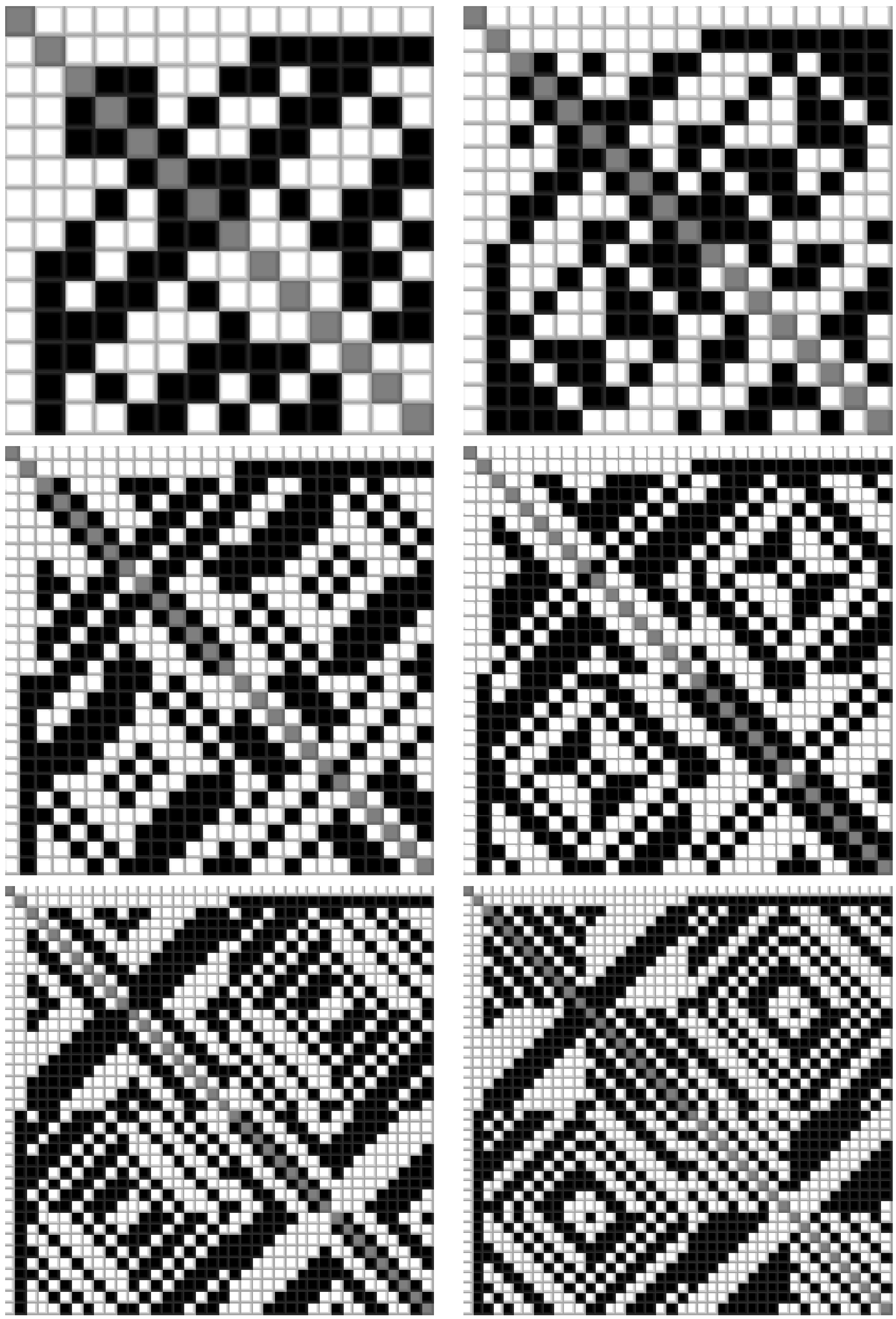

Figure 3: Matrices C_{14}, C_{18}, C_{26}, C_{30}, C_{38}, C_{42} of rich structure

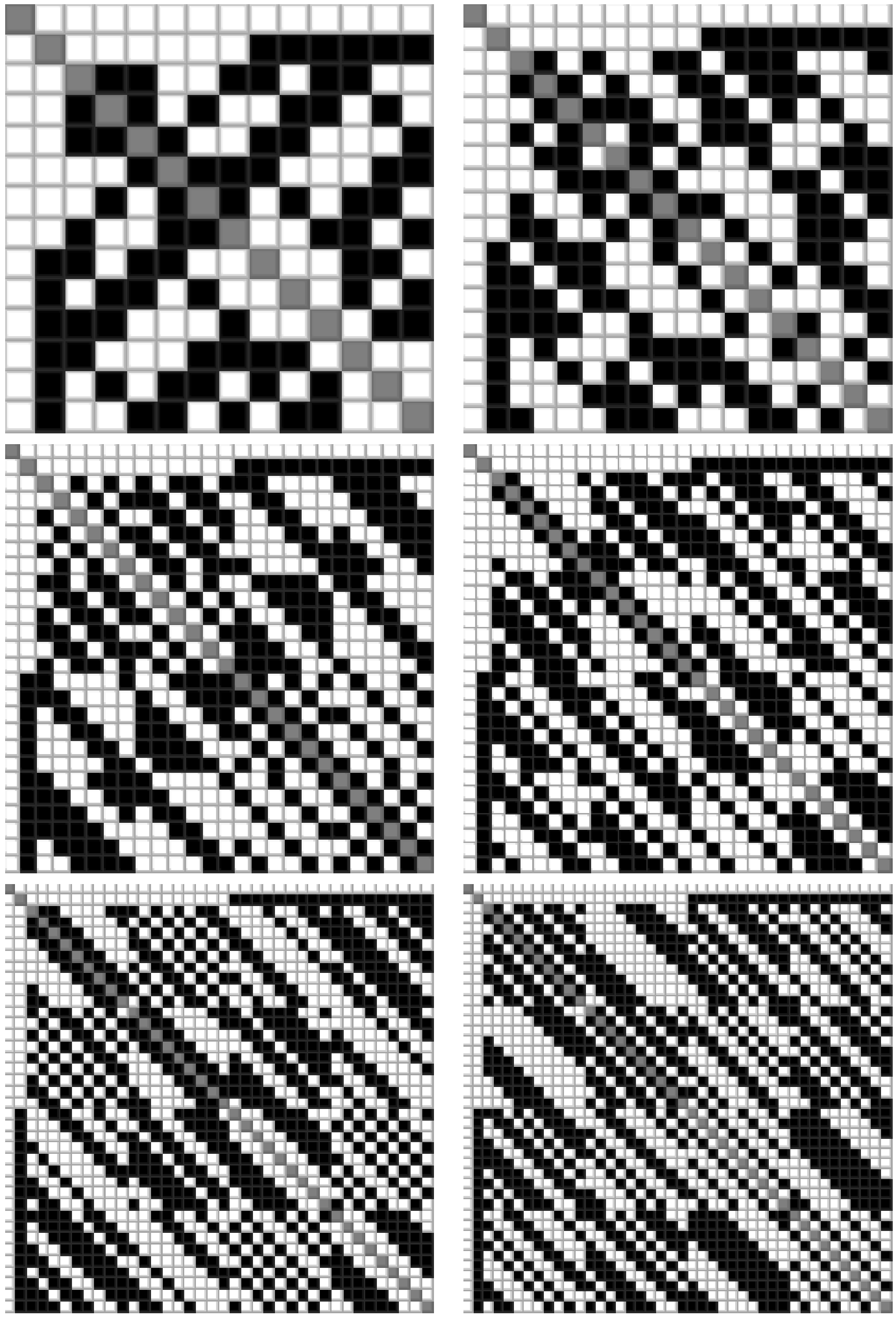

Figure 4: Matrices C_{14}, C_{18}, C_{26}, C_{30}, C_{38}, C_{42} of poor structure

10 The Frequency Selection of Cells

An Hadamard matrix selected by the frequency of its columns (the quantity of sign changes) reflect the well known Walsh functions [10].

We see here the other original type of selection connected with the frequency of cells A, B, C, D. The reasonable orders are $4^k + 2$. Matrix C_{18} (Fig. 3) has an "oscillated fast" block A, an "oscillated slowly" block B, and "otherwise even more slowly oscillated" blocks C and D.

The gross-set of matrices of other orders has cells selected by frequency too. This system could be used for the image processing.

11 Ornamental Destructions

In this paper we use black-and white pictures (*matrix portraits* [3]) to show the main structural properties.

We describe both family lines of orders $4t+2$ for odd and even t, and with our additional aim – to introduce the multicirculant solutions with specific features for a finite line of orders: 6, 26, 46, 66, 86, [106 does not exist].

The destruction of ornamental invariants can go much farther than the small difference between the odd and even constructions. The matrices of such *"singular points"* were found first by Rudy Mathon [4]. They were generalised by Jennifer Seberry and Albert Whiteman [9].

The new rich and poor Balonin-Seberry constructions of these matrices [2] are interesting as abstract mathematical objects. Their invariants play an important role: to be the labels of the critical orders when multicirculant matrices leave their simple presentation.

12 The Singular Points

We note that C_{10} belongs to the first specific class : the one border and one circulant matrix (core), does not exist. This order is given as the two border and two/four circulant case, the latest shown here.

A common property of finite family lines of orders is having a singular point inside: an example is the family with conference matrices of size 6, 26, 46 (singular point), 66, 86, [106 does not exist].

The first two conference matrices C_6, C_{26} of this family belong to the trivial and special cases with a simple picture solution. The next matrix C_{46} is a singular point of this matrix sequence: it has no simple solution, but it can be found as a matrix with rich structure: two borders and specific cells (more, than four such). While C_6 and C_{26} exist, the existence of C_{66}, C_{86} is an open problem. There is the set of standard classes they do not belong to: the matrix with one border and circulant core, the two-circulant matrix, and so on.

We say now that matrices of orders 66 and 86 do not exist as matrices with two borders and four circulant cells. Moreover, the presence of the singular point, 66, gives the principal and most important invariant of all such families.

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