Two Infinite Families of Symmetric Hadamard Matrices

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Dedicated to the Unforgettable Mirka Miller

Abstract

A construction method for orthogonal ±1 matrices based on a variation of the Williamson array called the *propus array*

$$\begin{array}{cccc} A & B & B & D \\ B & D & -A & -B \\ B & -A & -D & B \\ D & -B & B & -A \end{array}$$

gives symmetric propus-Hadamard matrices.

We show that for

- $q \equiv 1 \pmod 4$, a prime power, symmetric propus-Hadamard matrices exist for order $2(q+1)$; and
- $q \equiv 1 \pmod 4$, a prime power, and $\frac{1}{2}(q+1)$ a prime power or the order of the core of a symmetric conference matrix (this happens for $q = 89$) symmetric propus-type Hadamard matrices of order $4(2q+1)$ exist.

We give constructions to find symmetric propus-Hadamard matrices for 57 orders $4n$, $n < 200$ odd.

Keywords: Hadamard Matrices, D-optimal designs, conference matrices, propus construction, Williamson matrices; Cretan matrices; 05B20.

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1 Introduction

Hadamard matrices arise in statistics, signal processing, masking, compression, combinatorics, weaving, spectroscopy and other areas. They been studied extensively. Hadamard showed [13] the order of an Hadamard matrix must be 1, 2 or a multiple of 4. Many constructions for ±1 matrices and similar matrices such as Hadamard matrices, weighing matrices, conference matrices and D-optimal designs use skew and symmetric Hadamard matrices in their construction. For more details see Seberry and Yamada [29].

An Hadamard matrix of order n is an $n \times n$ matrix with elements ±1 such that $HH^\top = H^\top H = nI_n$, where I_n is the $n \times n$ identity matrix and $\top$ stands for transposition. A skew Hadamard matrix $H = I + S$ has $S^\top = -S$. For more details see the books and surveys of Jennifer Seberry (Wallis) and others [29, 33] cited in the bibliography.

Theorems of the type *for every odd integer n there exists a t dependent on n so that Hadamard, regular Hadamard, co-cyclic Hadamard and some full orthogonal designs exist for all orders $2^t n$, t integer* are known [36, 37, 38, 39]. A similar result for symmetric Hadamard and skew-Hadamard matrices has not yet been published but is conjectured.

The Propus construction is a construction method using orthogonal ±1 matrices, A, $B = C$, and D, where

$$AA^\top + 2BB^\top + DD^\top = \text{ constant } I,$$

I the identity matrix, called *propus matrices*, based on the array

$$\begin{array}{cccc} A & B & B & D \\ B & D & -A & -B \\ B & -A & -D & B \\ D & -B & B & -A \end{array}$$

to construct symmetric Hadamard matrices.

We give methods to find propus-Hadamard matrices: using Williamson matrices and D-optimal designs. These are then generalized to allow non-circulant symmetric matrices with the same aim to give symmetric Hadamard matrices.

We show that for

- $q \equiv 1 \pmod 4$, a prime power, the required matrices exist for order $t = \frac{1}{2}(q+1)$, and thus symmetric Hadamard matrices of order $2(q+1)$;
- $q \equiv 1 \pmod 4$, a prime power, and $\frac{1}{2}(q+1)$ a prime power or the order of the core of a symmetric conference matrix (this happens for $q = 89$) the required symmetric propus-type Hadamard matrices of order $4(2q+1)$ exist;

- $t \equiv 3 \pmod 4$, a prime, such that D-optimal designs, constructed using two circulant matrices, one of which must be circulant and symmetric, exist of order $2t$, then such symmetric Hadamard matrices exist for order $4t$.

- $4-\{t; s_1, s_2, s_3, s_4; \frac{\sum_{i=1}^{4} s_i(s_i-1)}{t-1}\}$ sds, $4t = a^2+b^2+c^2+d^2, a \equiv b = c \equiv d \equiv t \pmod 4$, $a = 2s_1 - t, b = 2s_2 - t$, $c = 2s_3 - t$, $d = 2s_4 - t$, where one of the supplementary difference sets is symmetric then such symmetric Hadamard matrices exist for order $4t$.

We note that appropriate *Williamson type* matrices may also be used to give propus-Hadamard matrices but do not pursue this avenue in this paper. There is also the possibility that this propus construction may lead to some insight into the existence or non-existence of symmetric conference matrices for some orders. We refer the interested reader to mathscinet.ru/catalogue/propus/.

1.1 Definitions and Basics

Two matrices X and Y of order n are said to be *amicable* if $XY^\top = YX^\top$.

We define the following classes of propus like matrices. We note that there are slight variations in the matrices which allow variant arrays and non-circulant matrices to be used to give symmetric Hadamard matrices, All propus like matrices A, $B = C$, D are ± 1 matrices of order n satisfy the *additive property*

$$AA^\top + 2BB^\top + DD^\top = 4nI_n, \tag{1}$$

I the identity matrix, J the matrix of all ones.

We consider the following classes of ± 1 of order n:

- *propus matrices*: four circulant symmetric ± 1 matrices, A, B, B, D of order n, satisfying the additive property (use P);

- *propus-type matrices*: four pairwise amicable ± 1 matrices, A, B, B, D of order n, $A^\top = A$, satisfying the additive property (use P);

- *generalized-propus matrices*: four pairwise commutative ± 1 matrices, A, B, B, D of order n, $A^\top = A$, which satisfy the additive property (use GP).

We use two types of arrays into which to plug the propus like matrices: the Propus array, P, or the generalized-propus array, GP. These can also be used with generalized matrices ([32]).

$$P = \begin{bmatrix} A & B & B & D \\ B & D & -A & -B \\ B & -A & -D & B \\ D & -B & B & -A \end{bmatrix} \quad \text{and} \quad GP = \begin{bmatrix} A & BR & BR & DR \\ BR & D^\top R & -A & -B^\top R \\ BR & -A & -D^\top R & B^\top R \\ DR & -B^\top R & B^\top R & -A. \end{bmatrix}.$$

2 Symmetric Propus-Hadamard Matrices

We first give the explicit statements of two well known theorem, Paley's Theorem [27], for the Legendre core Q, and Turyn's Theorem [30], in the form in which we will use them.

Theorem 1. **[Paley's Legendre Core [27]]** *Let p be a prime power, either $\equiv 1 \pmod 4$ or $\equiv 3 \pmod 4$ then there exists a matrix, Q, of order p with zero diagonal and other elements ± 1 satisfying $QQ^\top = (q+1)I - J$, Q is symmetric or skew-symmetric according as $p \equiv 1 \pmod 4$ or $p \equiv 3 \pmod 4$.*

Theorem 2. **[Turyn's Theorem [30]]** *Let $q \equiv 1 \pmod 4$ be a prime power then there are two symmetric matrices, P and S of order $\frac{1}{2}(q+1)$, satisfying $PP^\top + SS^\top = qI$: P has zero diagonal and other elements ± 1 and S elements ± 1.*

2.1 Propus-Hadamard Matrices from Williamson Matrices

Lemma 1. *Let $q \equiv 1 \pmod 4$, be a prime power, then propus matrices exist for orders $n = \frac{1}{2}(q+1)$ which give symmetric propus-Hadamard matrices of order $2(q+1)$.*

Proof. We note that for $q \equiv 1 \pmod 4$, a prime power, Turyn (Theorem 2 [30]) gave Williamson matrices, $X+I$, $X-I$, Y, Y, which are circulant and symmetric for orders $n = \frac{1}{2}(q+1)$. Then choosing

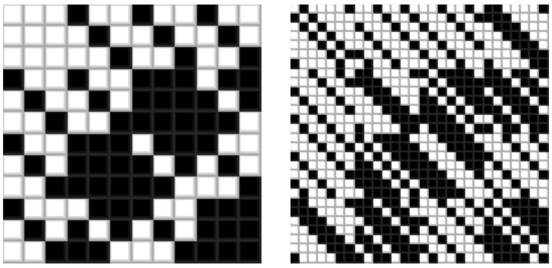
$$A = X + I,\ B = C = Y,\ D = X - I$$

gives the required propus-Hadamard matrices. □

We now have propus-Hadamard matrices for orders $4n$ where n is in

$$\{1, 3, [5], 7, 9, [13], 15, 19, 21, [25], 27, 31, 37, [41], 45, 49, 51, 55, 57, 59,$$
$$[61], [63], 67, 69, 75, 79, 81, [85], 87, 89, 91, 97, 99, 105, 111, 115, 117, 119, 121,$$
$$127, 129, 135, 139, 141, [145], 147, 157, 159, 169, 175, 177, [181], 187, 195, 199.\}$$

The cases written in square brackets [5],[13],[25],[41],[61],[63],[85],[113], [145],[181] arise when q is a prime power, however the Delsarte-Goethals-Seidel-Turyn construction means the required circulant matrices also exist for these prime powers.



(a) P12 ($q = 5; n = 3$) **(b)** P28 ($q = 13; n = 7$)

Figure 1: Propus-Hadamard matrices for orders $4q$

2.1.1 Propus matrices of small order and from q prime power

There are two trivial propus-Hadamard matrices of orders 12 and 20 based on $A = J$, $B = C = D = J - 2I$, for $n = 3$, and $A = Q + I$, $B = C = J - 2I$, $D = Q - I$ (Q constructed using Legendre symbols) for n=5.

2.2 Propus-Hadamard matrices from D-optimal designs

Lemma 2. *Let $n \equiv 3 \pmod 4$, be a prime, such that D-optimal designs, constructed using two circulant matrices, one of which is symmetric, exist for order $2n$. Then propus-Hadamard matrices exist for order $4n$.*

Djoković and Kotsireas in [22, 8] give D-optimal designs, constructed using two circulant matrices, for $n \in \{3, 5, 7, 9, 13, 15, 19, 21, 23, 25, 27, 31, 33, 37, 41, 43, 45, 49, 51, 55, 57, 59, 61, 63, 69, 73, 75, 77, 79, 85, 87, 91, 93, 97, 103, 113, 121, 131, 133, 145, 157, 181, 183\}$, $n < 200$. We are interested in those cases where the D-optimal design is constructed from two circulant matrices one of which must be symmetric.

Suppose D-optimal designs for orders $n \equiv 3 \pmod 4$, a prime, are constructed using two circulant matrices, X and Y. Suppose X is symmetric. Let $Q + I$ be the Paley matrix of order n. Then choosing

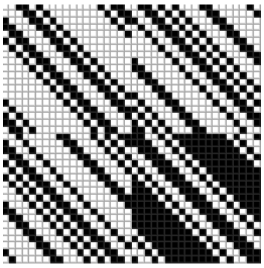
$$A = X, \quad B = C = Q + I, \quad D = Y,$$

to put in the array GP gives the required propus-Hadamard matrices.

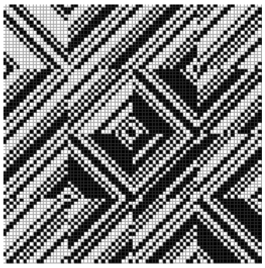
Hence we have propus-Hadamard matrices, constructed using D-optimal designs, for orders $4n$ where n is in

$$\{3, 7, 19, 31\}.$$

The results for $n = 19$ and 31 were given to us by Dragomir Djokovič.



(a) D38 ($n = 19$)



(b) GP76 ($n = 19$)

Figure 2: D-optimal designs for orders $2n$ propus-Hadamard matrices for orders $4n$

2.3 A Variation of a Theorem of Miyamoto

In Seberry and Yamada [29] one of Miyamoto's results [25] was reformulated so that symmetric Williamson-type matrices can be obtained. The results given here are due to Miyamoto, Seberry and Yamada.

Lemma 3 (Propus Variation). *Let U_i, V_j, $i, j = 1, 2, 3, 4$ be $(0, +1, -1)$ matrices of order n which satisfy*

(i) U_i, U_j, $i \neq j$ are pairwise amicable,

(ii) V_i, V_j, $i \neq j$ are pairwise amicable,

(iii) $U_i \pm V_i$, $(+1, -1)$ matrices, $i = 1, 2, 3, 4$,

(iv) the row sum of U_1 is 1, and the row sum of U_j, $i = 2, 3, 4$ is zero,

(v) $\sum_{i=1}^{4} U_i U_i^T = (2n+1)I - 2J$, $\sum_{i=1}^{4} V_i V_i^T = (2n+1)I$.

Let S_1, S_2, S_3, S_4 be four $(+1, -1)$-matrices of order $2n$ defined by

$$S_j = U_j \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + V_j \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

where $S_2 = S_3$.

Then there are 4 propus-Williamson type matrices of order $2n+1$. If U_i and V_i are symmetric, $i = 1, 2, 3, 4$ then the Williamson-type matrices are symmetric. Hence there is a symmetric propus-type Hadamard matrix of order $4(2n+1)$.

Proof. With S_1, S_2, S_3, S_4, as in the theorem enunciation the row sum of $S_1 = 2$ and of $S_i = 0$, $i = 2, 3, 4$. Now define

$$X_1 = \begin{bmatrix} 1 & -e_{2n} \\ -e_{2n}^T & S_1 \end{bmatrix} \quad \text{and} \quad X_i = \begin{bmatrix} 1 & e_{2n} \\ e_{2n}^T & S_i \end{bmatrix}, \quad i = 2, 3, 4.$$

First note that since U_i, U_j, $i \neq j$ and V_i, V_j, $i \neq j$ are pairwise amicable,

$$\begin{aligned} S_i S_j^T &= \left(U_i \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + V_i \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)\left(U_j^T \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + V_j^T \times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right) \\ &= U_i U_j^T \times \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} + V_i V_j^T \times \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \\ &= S_j S_i^T. \end{aligned}$$

(Note this relationship is valid if and only if conditions (i) and (ii) of the theorem are valid.)

$$\begin{aligned} \sum_{i=1}^{4} S_i S_i^T &= \sum_{i=1}^{4} U_i U_i^T \times \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} + \sum_{i=1}^{4} V_i V_i^T \times \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \\ &= 2\begin{bmatrix} 2(2n+1)I - 2J & -2J \\ -2J & 2(2n+1)I - 2J \end{bmatrix} \\ &= 4(2n+1)I_{2n} - 4J_{2n} \end{aligned}$$

Next we observe

$$X_1 X_i^T = \begin{bmatrix} 1 - 2n & e_{2n} \\ e_{2n}^T & -J + S_1 S_i^T \end{bmatrix} = X_i X_1^T \qquad i = 2, 3, 4,$$

and

$$X_i X_j^T = \begin{bmatrix} 1 + 2n & e_{2n} \\ e_{2n}^T & J + S_i S_j^T \end{bmatrix} = X_j X_i^T \qquad i \neq j, \quad i, j = 2, 3, 4.$$

Further

$$\begin{aligned} \sum_{i=1}^{4} X_i X_i^T &= \begin{bmatrix} 1 + 2n & -3e_{2n} \\ -3e_{2n}^T & J + S_1 S_1^T \end{bmatrix} + \sum_{i=2}^{4} \begin{bmatrix} 1 + 2n & e_{2n} \\ e_{2n}^T & J + S_i S_i^T \end{bmatrix} \\ &= \begin{bmatrix} 4(2n+1) & 0 \\ 0 & 4J + 4(2n+1)I - 4J \end{bmatrix}. \end{aligned}$$

Thus we have shown that X_1, X_2, X_3, X_4 are pairwise amicable, symmetric Williamson type matrices of order $2n + 1$, where $X_2 = X_3$. These can be used as in (ii) of Theorem using the additive property to obtain the required symmetric propus Hadamard matrix of order $(4(2n + 1)$. □

Many powerful corollaries arose and new results were obtained by making suitable choices in the theorem. We choose X_1, X_2, X_3, X_4 to ensure that the propus construction can be used to form symmetric Hadamard matrices of order $4(2n + 1)$.

From Paley's theorem (Corollary 1) for $p \equiv 3 \pmod 4$ we use the backcirculant or type 1, symmetric matrices QR and R instead of Q and I; whereas for $p \equiv 1 \pmod 4$ we use the symmetric Paley core Q. If p is a prime power $\equiv 3 \pmod 4$ we set $U_1 = I$, $U_2 = U_3 = QR$, $U_4 = 0$ of order p, and if p is a prime power $\equiv 1 \pmod 4$, we set $U_1 = I$, $U_2 = U_3 = Q$, $U_4 = 0$ of order p. Hence $\sum_{k=1}^{4} U_k U_k^\top = (q+2)I - 2J$.

From Turyn's result (Corollary 2) we set, for $p \equiv 1 \pmod 4$ $V_1 = P$, $V_2 = V_3 = I$ and $V_4 = S$, and for $p \equiv 3 \pmod 4$, $V_1 = P$, $V_2 = V_3 = R$ and $V_4 = S$, so $\sum_{k=1}^{4} V_k V_k^\top = (q+2)I$.

Hence we have:

Corollary 1. *Let $q \equiv 1 \pmod 4$ be a prime power and $\frac{1}{2}(q+1)$ be a prime power or the order of the core of a symmetric conference matrix (this happens for $q = 89$). Then there exist symmetric Williamson type matrices of order $2q+1$ and a symmetric propus-type Hadamard matrix of order $4(2q+1)$.*

Using $q = 5$ and $q = 41$ gives the previously unresolved cases for 11 and 83.

2.3.1 Three Equal

The two starting Hadamard matrices of orders 12 and 28 based on the skew Paley core $B = C = D = Q + I$ (constructed using Legendre symbols) are unique and finite because $12 = 3^2 + 1^2 + 1^2 + 1^2$ and $28 = 5^2 + 1^2 + 1^2 + 1^2$ and these are the only orders for which a symmetric circulant A can exist with $B = C = D$.

3 Propus-Hadamard matrices from conference matrices: even order matrices

A powerful method to construct propus-Hadamard matrices for n even is using conference matrices.

Lemma 4. *Suppose M is a conference matrix of order $n \equiv 2 \pmod 4$. Then $MM^\top = M^\top M = (n-1)I$, where I is the identity matrix and $M^\top = M$. Then using $A = M + I$, $B = C = M - I$, $D = M + I$ gives a propus-Hadamard matrix of order $4n$.*

We use the conference matrix orders from [1] and so have propus-Hadamard matrices of orders $4n$ where $n \in$

$$\{6, 10, 14, 18, 26, 30, 38, 42, 46, 50, 54, 62, 74, 82, 90, 98\}.$$

Conference matrices can be constructed using made two circulant matrices A and B of order n where both A and B are symmetric.

Then using the matrices $A+I$, $B=C$ and $D=A-I$ in P gives the required construction.

The conference matrices can also be made from two circulant matrices A and B of order n where both A and B are symmetric. However here we use $A+I$, $BR=CR$ and $D=A-I$ in P to obtain the required construction. There is another variant of this family which uses the symmetric Paley cores $A=Q+I$, $D=Q-I$ (constructed using Legendre symbols) and one circulant matrix of maximal determinant $B=C=Y$.

4 Conclusion and Future Work

Using the results of Lemma 1 and Corollary 1 and the symmetric propus-Hadamard matrices of Di Matteo, Djoković, and Kotsireas given in [4], we see that the unresolved cases for symmetric propus-Hadamard matrices for orders $4n$, $n<200$ odd, are where $n \in$

$$\{17,23,29,33,35,39,47,53,65,71,73,77,93,95,97,99, 101,103,107,109,113,125,131,133,137,143,149,151,153,155, 161,163,165,167,171,173,179,183,185,189,191,193,197.\}$$

There are many constructions and variations of the propus theme to be explored in future research. Visualizing the propus construction gives aesthetically pleasing examples of propus-Hadamard matrices. The visualization also makes the construction method clearer. There is the possibility that these visualizations may be used for quilting.

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